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**Abstract**

**Full Text**

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## INTERPOLATION OF SPACES OF DIFFERENTIABLE FUNCTIONS DEFINED IN AN $n$ -DIMENSIONAL DOMAIN

*(Presented by Academician I. M. Vinogradov on 26 IX 1968)*

1. In the paper <sup>(1)</sup> S. M. Nikol'skii proved interpolation theorems for the spaces  $S_p^{r_i} H(E^n)$  ( $i = 1, \dots, N$ ) (where  $\mathbf{r}_i$  are positive vectors, i.e.  $r_j^i > 0$ ).

Subsequently, in <sup>(4)</sup>, P. I. Lizorkin and S. M. Nikol'skii extended these results to the well-known spaces  $S_p^{r_i} L(E^n)$  ( $i = 1, \dots, N$ ) ( $\mathbf{r}_i$  are positive vectors). T. I. Amanov in <sup>(15)</sup> proved interpolation theorems for the spaces  $S_{p,\theta}^{r_i} B(E^n)$  ( $i = 1, \dots, N$ ) (where  $\mathbf{r}_i$  are positive vectors). In <sup>(9)</sup> Grisvard proved interpolation theorems for the spaces  $S_{p_1}^{r_1} B(E^n)$ ,  $S_{p_2}^{r_2} B(E^n)$  (where  $\mathbf{r}_1$  and  $\mathbf{r}_2$  are positive vectors). In the present paper we define spaces of functions with dominating mixed derivatives, defined in an  $n$ -dimensional (not necessarily bounded) domain  $\Omega$ , i.e. spaces of the type  $S_{p,\theta}^r B(\Omega)$ ,  $S_p^r W(\Omega)$ . For definitions of these spaces when  $\Omega = E^n$ , see <sup>(1, 2, 4, 8, 10, 12, 14)</sup>.

In the present paper interpolation theorems are announced for the spaces  $S_{p_i, \theta_i}^{r_i} B(\Omega)$  ( $i = 1, \dots, N$ ) and  $S_{p_i}^{r_i} W(\Omega)$  ( $i = 1, \dots, N$ ) (where  $\mathbf{r}_i$  are nonnegative vectors, i.e.  $r_j^i \geq 0$ ).

For  $\theta_i = \infty$  ( $i = 1, \dots, N$ ) we obtain interpolation theorems for the closure of the set of smooth functions in the spaces  $S_{p_i}^{r_i} H(\Omega)$  ( $i = 1, \dots, N$ ), extending the corresponding results <sup>(1)</sup> of S. M. Nikol'skii to different  $p_i$  and to the domain  $\Omega$  (without assuming strict positivity of the vectors  $\mathbf{r}_i$  ( $i = 1, \dots, N$ )).

The results of <sup>(1, 2, 4, 9, 15)</sup> are extended to vectors  $\mathbf{r}_i$  ( $i = 1, \dots, N$ ) with non-negative components, to different  $1 \leq p_i \leq \theta_i \leq \infty$  ( $i = 1, \dots, N$ ), and to the domain  $\Omega$ .

From the interpolation theorems that are formulated in the present paper there follows a series of new results also for the well-known spaces  $W_p^r$ ,  $H_p^r$  and  $B_p^r$  (see <sup>(3, 5, 6)</sup>). In the present paper domains subject to restrictions 1) and 2) will be considered. In the present paper we shall consider domains from the class of domains with  $(\mathbf{h})$ , i.e. from the set of domains attainable "from within" by a parallelepiped whose edges are parallel to the coordinate axes and are respectively equal to the components of the vector  $\mathbf{h} = (h_1, \dots, h_n)$ . Also considered are

domains from the classes of domains  $\bigcap_{i=1}^n R^{(i)}$ , which are defined and studied in the work of V. P. Il' in <sup>(5)</sup>.

2. Let  $e_n$  be the set of natural numbers  $1, \dots, n$ , i.e.  $e_n = \{1, \dots, n\}$ . If  $\mathbf{K} = (k_1, \dots, k_n)$  is some vector, then by  $\mathbf{K}^e = (k_1^e, \dots, k_n^e)$  (where  $e$  is any subset of the set  $e_n$ ) we shall denote the vector with components  $k_i^e = k_i$  for  $i \in e$  and  $k_i = 0$  for  $i \notin e$ . By  $e_k$  we denote the support of the vector  $\mathbf{K}$ , i.e. the smallest subset  $e$  for which  $\mathbf{K}^e = \mathbf{K}$  (in other words,  $e_k$  is the set of indices of the nonzero components of the vector  $\mathbf{K}$ ; for more detail see <sup>(1)</sup>).

**Definition 1.** We shall say that  $f \in S_{p,\theta}^r B(\Omega, \mathbf{h})$ , where  $1 \leq p \leq \theta \leq \infty$ ,  $\mathbf{r} = (r_1, \dots, r_n)$ ,  $r_i \geq 0$ ,  $e_r \neq \emptyset$ , if  $f \in L_p(\Omega)$  and there exist

in  $\Omega$  generalized derivatives in the sense of Sobolev <sup>(7)</sup> (for all  $e \subseteq e_r$ ),

$$D^{\bar{r}^e} f(x) \equiv \frac{\partial^{|\bar{r}^e|} f(x)}{\partial x_1^{\bar{r}_1^e} \dots \partial x_n^{\bar{r}_n^e}}, \quad |\bar{r}^e| = \sum_{i=1}^n \bar{r}_i^e$$

(where  $\bar{r}_i$  is the greatest integer  $< r_i$ ), satisfying the conditions

$$\|f\|_{\mathcal{L}_{p,\theta}^{r^e}(\Omega, \mathbf{h})} = \left( \int_{\square_{\mathbf{h}}(0)} \dots \int \left\| \Delta^{2\bar{\omega}^e} \left( \frac{t}{2} \right) D^{\bar{r}^e} f \right\|_{L_p(\Omega_{t^e, 2\bar{\omega}^e})}^{\theta} \prod_{i \in e} \frac{dt_i}{|t_i|^{1+\theta\alpha_i}} \right)^{1/\theta} < \infty,$$

where  $\bar{\omega} = (1, \dots, 1)$ ;  $\alpha_i = r_i - \bar{r}_i$ ;  $\Omega_{t^e, 2\bar{\omega}^e}^* = \bigcap_{i \in e} \Omega_{t_i, k_i}$ ; and  $\Delta^k(t)f = \Delta_1^{k_1}(t_1) \dots \Delta_n^{k_n}(t_n)f$ ;  $\Delta_i^{k_i}(t_i)f$  is the finite difference of order  $k_i$  with respect to the variable  $x_i$  with step  $t_i$ ,  $\square_{\mathbf{h}}(0) = (-h_1, h_1) \times \dots \times (-h_n, h_n)$ .

We define the norm in this space as follows:

$$\|f\|_{S_{p,\theta}^r B(\Omega, \mathbf{h})} = \|f\|_{L_p(\Omega)} + \sum_{e \subseteq e_r} \|f\|_{\mathcal{L}_{p,\theta}^{r^e}(\Omega, \mathbf{h})}.$$

The spaces  $S_{p,\theta}^r B(\Omega, \mathbf{h})$  for  $\theta = \infty$  coincide with the spaces  $S_{pH}^r(\Omega, \mathbf{h})$ , defined for  $\Omega = E^n$  in <sup>(1)</sup>.

**Definition 2.** We shall say that  $f \in S_{pW}^r(\Omega, \mathbf{h})$  (where  $r = (r_1, \dots, r_n)$  is a nonnegative vector with support  $e_r \neq \emptyset$ ) if  $f \in L_p(\Omega)$  and for all  $e \subseteq e_r$  there exist in  $\Omega$  generalized derivatives  $D^{\tilde{r}^e} f$  (where  $\tilde{r} = (\tilde{r}_1, \dots, \tilde{r}_n)$ ,  $\tilde{r}_i$  is the integer part of  $r_i$ ), satisfying the conditions

$$\|f\|_{L_p^{r^e}(\Omega, \mathbf{h})} = \begin{cases} \|D^{r^e} f\|_{L_p(\Omega)} < \infty, & \text{if } e^* \equiv e \cdot e_r^* = \emptyset, \\ \left( \int_{\square_{\mathbf{h}e^*(0)}} \dots \int \|\Delta^{\bar{e}^*}(t) D^{\tilde{r}^e} f\|_{L_p(\Omega_{t^*}^{e^*, \omega})}^p \prod_{i \in e^*} \frac{dt_i}{|t_i|^{1+n\alpha_i}} \right)^{1/p} < \infty, & \text{if } e^* \neq \emptyset, \end{cases}$$

where  $\alpha_i = r_i - \tilde{r}_i$ ,  $e_r^*$  is the set of indices of the noninteger components of the vector  $r$ .

We define the norm in these spaces in the following way:

$$\|f\|_{S_{pW}^r(\Omega, \mathbf{h})} = \|f\|_{L_p(\Omega)} + \sum_{e \subseteq e_r} \|f\|_{L_p^{r^e}(\Omega, \mathbf{h})}.$$

The study of these spaces is the subject of the works <sup>(2, 4, 12, 14)</sup>.

The closure of the set of smooth functions in the corresponding norm will be denoted by  $\dot{S}_{p,\theta}^r B(\Omega, \mathbf{h})$ ,  $\dot{S}_{pW}^r(\Omega, \mathbf{h})$ .

**3. Theorem 1.** *Let  $f(x)$  belong simultaneously to all spaces*

$$S_{p_i, \theta_i}^{r_i} B(\Omega, \mathbf{h}) \quad (i = 1, \dots, N),$$

where  $1 \leq p_i \leq \theta_i \leq \infty$  ( $i = 1, \dots, N$ ),  $r^i$  are nonnegative vectors with supports  $e_{r^i} \neq \emptyset$  ( $i = 1, \dots, N$ ), and the domain  $\Omega \in c(\mathbf{h})$  (see (5)). Suppose further that  $\lambda_i \geq 0$  ( $i = 1, \dots, N$ ) and

$$\sum_1^N \lambda_i = 1.$$

Then, for any integer vector  $\vec{v} = (v_1, \dots, v_n)$  ( $v_j \geq 0$  integers) with support

$$e_{\vec{v}} \subseteq \bigcup_{i=1}^N e_{r^i} \quad \text{and} \quad v_j < \sum_{i=1}^N r_j^i \lambda_i \quad (j = 1, \dots, n),$$

the function  $D^{\vec{v}} f$

belongs to  $L_p(\Omega)$  (where  $\frac{1}{p} = \sum_{i=1}^N \frac{\lambda_i}{p_i}$ ), and the following inequality holds

\*  $\Omega_{t_i, k_i}$  is the set of those points  $x = (x_1, \dots, x_n)$  of the domain  $\Omega$  such that the corresponding points  $(x_1, \dots, x_n)$

$$\|D^{\vec{v}}f\|_{L_p(\Omega)} \leq c \prod_{i=1}^N \|f\|_{S_{p_i, \theta_i}^{\lambda_i} B(\Omega, h)} \leq c \sum_{i=1}^N \|f\|_{S_{p_i, \theta_i}^{r_i} B(\Omega, h)}, \quad (1)$$

where  $c$  is a constant independent of  $f$ .

**Theorem 2.** Let the conditions of Theorem 1 be satisfied and let  $\Gamma^m$  be an  $m$ -dimensional ( $1 \leq m \leq n-1$ ) surface belonging to the domain  $\Omega$  (where  $\Omega \in c(h)$ ; see (4)) and defined by the equations:  $x_1 = x_1, \dots, x_m = x_m, x_{m+1} = \varphi_{m+1}(x_1, \dots, x_m), \dots, x_n = \varphi_n(x_1, \dots, x_m)$ , where  $\varphi_j(x_1, \dots, x_m)$  ( $j = m+1, \dots, n$ ) are continuous functions having bounded first derivatives.

Then, if  $e_m = \{m+1, \dots, n\} \subseteq \bigcup_{i=1}^N e_{r_i}$  and  $\vec{v} = (v_1, \dots, v_n)$  is an integer vector with components

$$0 \leq v_j < \sum_{i=1}^N r_j^i \lambda_i \quad (j = 1, \dots, m),$$

$$0 \leq v_j < \sum_{i=1}^N r_j^i \lambda_i - \frac{1}{p} \quad (j = m+1, \dots, n), \quad \left( \frac{1}{p} = \sum_{i=1}^N \frac{\lambda_i}{p_i} \right),$$

then the function  $D^{\vec{v}}f|_{\Gamma^m}$  belongs to  $L_p(\Omega)$ , and the inequality

$$\|D^{\vec{v}}f|_{\Gamma^m}\|_{L_p(\Gamma^m)} \leq c \prod_{i=1}^N \|f\|_{S_{p_i, \theta_i}^{\lambda_i} B(\Omega, h)} \leq c \sum_{i=1}^N \|f\|_{S_{p_i, \theta_i}^{r_i} B(\Omega, h)} \quad (2)$$

holds, where  $c$  is a constant independent of  $f$ .

**Theorem 3.** If in Theorems 1 and 2 one assumes that  $\bigcup_{i=1}^N e_{r_i} = e_n$ ,  $q \geq p$  (where  $\frac{1}{p} = \sum_{i=1}^N \frac{\lambda_i}{p_i}$ ) and

$$\kappa_j = \sum_{i=1}^N r_j^i \lambda_i - v_j - \left( \frac{1}{p} - \frac{1}{q} \right) > 0 \quad (j = 1, \dots, m),$$

$$\kappa_j = \sum_{i=1}^N r_j^i \lambda_i - v_j - \frac{1}{p} > 0 \quad (j = m+1, \dots, n), \quad (3)$$

then  $D^{\vec{v}}f \in L_q(\Gamma^m)$ , and an inequality analogous to inequalities (1) and (2) is valid, with the difference that on the left-hand side of these inequalities there will be norms in  $L_q(\Gamma^m)$  ( $\Gamma^m \equiv \Omega$ ).

Analogous theorems are also valid for the spaces  $S_{p_i}^{r_i} \widetilde{W}(\Omega, h)$ .

**Theorem 4.** Let  $f(x)$  belong to all the spaces  $S_{p_i, \theta_i}^{r_i} \widetilde{B}(\Omega, h)$  ( $i = 1, \dots, N$ ), where  $1 \leq p_i \leq \theta_i \leq \infty$  ( $i = 1, \dots, N$ ),  $e_{r_i} = e_n$  ( $i = 1, \dots, N$ ) (i.e.  $r_j^i > 0$  and  $\Omega \in \bigcap_{j=1}^n R^{(j)}$  (see (5))).

Suppose further that  $\lambda_i \geq 0$  ( $i = 1, \dots, N$ ),  $\sum_1^N \lambda_i = 1$  and  $\frac{1}{p} = \sum_1^N \frac{\lambda_i}{p_i}$ ,  $\frac{1}{\theta} = \sum_1^N \frac{\lambda_i}{\theta_i}$ .

Let  $\kappa_j > 0$  ( $j = 1, \dots, n$ ) (see (3)), and let  $\Omega^m$  be the intersection of the domain  $\Omega$  with the hyperplane  $x_{m+1} = \text{const}, \dots, x_n = \text{const}$ .

Then  $D^{\vec{v}} f \in S_{q, \sigma}^{\vec{\rho}} \widetilde{B}(\Omega^m, h)$ , where  $q \geq p$ ,  $\sigma \geq \theta$  and  $\vec{\rho} = (\rho_1, \dots, \rho_m)$ ,  $0 \leq \rho_j \leq \kappa_j$  ( $j = 1, \dots, m$ ), and the inequalities

$$\|D^{\vec{v}} f\|_{S_{q, \sigma}^{\vec{\rho}} \widetilde{B}(\Omega^m, h)} \leq c \prod_{i=1}^N \|f\|_{S_{p_i, \theta_i}^{r_i} \widetilde{B}(\Omega, h)}^{\lambda_i} \leq c \sum_{i=1}^N \|f\|_{S_{p_i, \theta_i}^{r_i} \widetilde{B}(\Omega, h)}$$

hold, where  $c$  is a constant independent of  $f$ .

**Remark to Theorem 4.** The theorem remains valid if the vectors  $\mathbf{r}^i$  are replaced by the vectors  $\mathbf{l}, (0, \dots, r_i, \dots, 0) + \mathbf{l}$ ; ( $i = 1, \dots, n$ ), where  $\mathbf{l}$  is an arbitrary vector with nonnegative components; however, in this case it is necessary to assume that  $p_i = p$ ,  $\theta_i = \theta$  ( $i = 1, \dots, n$ ). The advantage of this remark is that vectors of this type may also contain zero components.

**Theorem 5.** If in Theorem 4 it is assumed that  $\vec{\rho} = (\rho_1, \dots, \rho_m)$  and  $\rho_j < \chi_j$  ( $j = 1, \dots, m$ ), then the assertions of this theorem remain valid also when

$$\bigcup_{i=1}^N e_{r_i} = e_n, \quad e_{r_i} \neq \emptyset \quad (i = 1, \dots, N).$$

**Corollary.** Let  $f \in B_{p, \vec{\theta}}^r$  ( $\mathbf{p} = (p_1, \dots, p_n)$ ,  $\vec{\theta} = (\theta_1, \dots, \theta_n)$ ),

$$1 \leq p_i \leq \theta_i \leq \infty \quad (i = 1, \dots, n), \quad \lambda_i > 0, \quad \sum_1^n \lambda_i = 1;$$

then  $f(x)$  belongs to the space  $S_{p, \theta}^{\vec{\rho}} B$  ( $\vec{\rho} = (\rho_1, \dots, \rho_n)$ ,  $\rho_i < \lambda_i r_i$  ( $i = 1, \dots, n$ )), and the inequalities

$$\|f\|_{S_{p, \theta}^{\vec{\rho}} B} \leq c \prod_{i=1}^n \|f\|_{B_{p_i, \theta_i}^{r_i}}^{\lambda_i} \leq c \|f\|_{B_{p, \vec{\theta}}^r} \left( \frac{1}{p} = \sum_{i=1}^n \frac{\lambda_i}{p_i}; \frac{1}{\theta} = \sum_{i=1}^n \frac{\lambda_i}{\theta_i} \right),$$

hold, where  $c$  is a constant independent of  $f$ .

Theorems analogous to Theorem 5 are also valid for the spaces

$$S_{p_i}^{r_i} W(\Omega) \quad (i = 1, \dots, n).$$

Under certain additional conditions on the domain  $\Omega$ , it is proved that the sets of sufficiently smooth finite functions in  $E^n$  are dense, respectively, in the spaces  $S_{p,\theta}^r B(\Omega)$  and  $S_p^r W(\Omega)$ . In particular, such domains include all bounded domains  $\Omega$  from  $\bigcap_{i=1}^n R^{(i)}$ , the domain  $E^n$ , and  $\{x; x_i > 0, i \in e \subset e_n\}$ . The equivalent normings of the spaces  $S_{p,\theta}^r B(\Omega)$  (see (4), § 4), which occurred for  $\Omega = E^n$ , remain valid also for  $\Omega \in \bigcap_{i=1}^n R^{(i)}$ .

The indicated results are proved by the method of integral representations (see (5,7,11,14)).

We note that some analogous results (for example, Theorem 4 and the remark to this theorem) for  $\Omega = E^n$ , under a more general assumption on summability, i.e., for spaces of functions with mixed norms, are contained in a joint work of O. V. Besov and the author.

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