SYSTEMS OF SINGULAR INTEGRAL EQUATIONS IN WEIGHTED \(L_p\) SPACES

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Abstract

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SYSTEMS OF SINGULAR INTEGRAL EQUATIONS IN WEIGHTED L_p SPACES

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1. Let Γ be a simple smooth closed oriented curve in the plane, surrounding the point $\lambda = 0$; let $t_1, \dots, t_n$ be certain points on Γ; and let $p, \beta_1, \dots, \beta_n$ be real numbers satisfying the relations

$$1 < p < \infty; \qquad -1 < \beta_k < p-1 \quad (k = 1, \dots, n). \tag{1}$$

By $L_p(\Gamma, \rho)$ we denote the space L_p on Γ with weight $\rho(t) = \prod |t - t_k|^{\beta_k}$, and by $L_p^m(\Gamma, \rho)$ the space of vector-functions $\varphi = (\varphi_1, \dots, \varphi_m)$ with components $\varphi_j \in L_p(\Gamma, \rho)$.

Consider in $L_p^m(\Gamma, \rho)$ the matrix singular integral operators

$$(K\varphi)(t) = C(t)\varphi(t) + D(t)(S\varphi)(t), \qquad (L\varphi)(t) = C(t)\varphi(t) + (SD\varphi)(t),$$

where $C(t)$ and $D(t)$ are matrix-functions ($t \in \Gamma$), and S is the matrix operator of singular integration

$$(S\varphi)(t) = \frac{1}{\pi i} \int_\Gamma \frac{\varphi(\tau)}{\tau - t}\, d\tau.$$

The operators K and L, in the case of continuous coefficients C and D, are Φ-operators if and only if the determinant $\det(C+D)(C-D)$ does not vanish anywhere on Γ. In the case of piecewise-continuous coefficients this condition, while remaining necessary, ceases to be sufficient for K and L to be Φ-operators.

In the present note, for the case of piecewise-continuous coefficients $C(t)$ and $D(t)$, necessary and sufficient conditions are established under which K and L are Φ-operators. The results obtained are a generalization to the case of systems of equations of the results of the authors ([1]) (see also ([2–4])).

The operators K and L, under the condition that the matrices C and D are composed of piecewise Hölder functions satisfying a Lipschitz condition in one-sided neighborhoods of the points of discontinuity, were studied by N. P. Vekua ([5]). In particular, he obtained sufficient conditions under which the operators K and L are Φ-operators in certain classes of Hölder functions. Under the same restrictions on the coefficients C and D, the operators K and L in the spaces $L_p^m(\Gamma, \rho)$ were considered by B. V. Khvedelidze ([6]), who also obtained certain sufficient conditions under which K and L are Φ-operators. In the paper of E. Shamir ([7]), operators of the type K and L are considered for the case when the contour Γ coincides with the real axis, while the matrices C and D are piecewise differentiable and satisfy certain conditions at infinity. Under these assumptions and the additional assumption of the nonsingularity of the matrices $C + D$ and $C - D$, he found necessary and sufficient conditions for the operators K and L to be Φ-operators in the spaces L_p.

Sufficient conditions for K and L to be Φ-operators in $L_2^m(\Gamma)$, in the case of bounded measurable coefficients C and D, were obtained by I. B. Simonenko ([8]).

We note that the method of investigation presented below differs from the methods of the authors listed above.

2. Denote by $\Lambda_m(\Gamma)$ the set of all matrix-functions of order m that are piecewise continuous and continuous from the left on Γ. Let $G(t) \in \Lambda_m(\Gamma)$; $t_1, \dots, t_n \in \Gamma$ are all points of discontinuity of the matrix G, and $\omega = (p, \beta_1, \dots, \beta_n)$ is a vector whose coordinates satisfy relations (1).

To the matrix-function $G(t)$ and the vector ω we assign the continuous matrix curve $V(G, \omega)$, obtained by adding to the set of values $G(t)$ n matrix arcs $W_k(\mu)$ $(0 \le \mu \le 1)$, where*

$$W_k(\mu) = \frac{e^{i\mu\theta_k} \sin(1-\mu)\theta_k}{\sin\theta_k} G(t_k) + \frac{e^{i(\mu-1)\theta_k} \sin\mu\theta_k}{\sin\theta_k} G(t_k + 0)$$

$$\left(\theta_k = \pi - \frac{2\pi(1+\beta_k)}{p}\right).$$

We shall call the matrix-function $G(t)$ ω-**regular** if the curve $\det V(G, \omega)$ does not pass through zero. The continuous closed curve $\det V$ in the plane is oriented so that, at points of continuity of the matrix $G(t)$, motion along the curve $\det V(G, \omega)$ is determined by the motion of t along Γ in the positive direction, while along the additional arcs it corresponds to the change of μ from 0 to 1.

Introduce the following notation: $C(t) + D(t) = A(t)$, $C(t) - D(t) = B(t)$, $I + S = 2P$, $I - S = 2Q$, where I is the identity operator; then the operators K and L may be written in the form $K = AP + BQ$, $L = PA + QB$.

Theorem 1. Let A and $B \in \Lambda_m$; $t_1, \dots, t_n \in \Gamma$ are all points of discontinuity of the matrices A and B; $\rho = \prod |t - t_k|^{\beta_k}$, where $-1 < \beta_k < p - 1$ $(1 < p < \infty)$. In order that the operator $K = AP + BQ$ $(L = PA + QB)$ be a Φ-operator in $L_p^m(\Gamma, \rho)$, it is necessary and sufficient that the following two conditions be fulfilled:

$$\alpha) \quad \det B(t\pm 0)\neq 0 \quad \text{for all } t\in\Gamma;$$

$$\beta) \quad \text{the matrix } B^{-1}A\ (AB^{-1}) \text{ is } \omega\text{-regular.}$$

If conditions $\alpha)$ and $\beta)$ are fulfilled, then the index of the operator $K(L)$ is determined by the equality

$$\varkappa = -\operatorname{ind}\det V(B^{-1}A,\omega) \quad (\varkappa = -\operatorname{ind}\det V(AB^{-1},\omega)). \tag{*}$$

The proof of this theorem is based on two lemmas.

Lemma 1. Let $M(t)$ and $N(t)$ be regular matrix-functions of order m, continuous on Γ, and let $X(t)$ be a bounded measurable matrix-function of the same order. In order that the operator $MXNP+Q$ $(PMXN+Q)$ be a Φ-operator in $L_p^m(\Gamma,\rho)$, it is necessary and sufficient that the operator $XP+Q$ $(PX+Q)$ be a Φ-operator in $L_p^m(\Gamma,\rho)$.

Proof. Since, for any continuous matrix-function $C(t)$, the operators $CP-PC$ and $CQ-QC$ are completely continuous in $L_p^m(\Gamma,\rho)$, and the operators $PN+QM^{-1}$ and $MP+N^{-1}Q$ are Φ-operators, the assertion of the lemma follows from the following easily verified equalities:

$$MXNP+Q = M\big[(XP+Q)(PN+QM^{-1})+X(NP-PN)+$$

$$+M^{-1}Q-QM^{-1}\big],$$

$$PMXN+Q = \big[(MP+N^{-1}Q)(PX+Q)+(PM-MP)X+ \tag{2}$$

$$+QN^{-1}-N^{-1}Q\big]N.$$

Lemma 2. Every piecewise-continuous matrix-function $G(t)$ $(t\in\Gamma)$, satisfying at each point of discontinuity t_k $(k=1,\dots,n)$ the condition $\det G(t_k\pm 0)\neq 0$, can be represented in the form

$$G(t) = M(t)X(t)N(t), \tag{3}$$

* If $2\beta_k = p-2$, then as $W_k(\mu)$ one takes the matrix segment $W_k = G(t_k)(1-\mu)+G(t_k+0)\mu$.

where M and N are continuous (nonsingular) matrix-functions, and X is a piecewise-continuous triangular matrix-function.

Proof. Choose nonsingular constant matrices N_k so that $N_{kG}^{-1}(t_k+0)G(t_k-0)N_k^{-1}$ are upper triangular matrices. Let $N(t)$ be a continuous nonsingular matrix-function on Γ, subject to the sole condition $N(t_k)=N_k$ $(k=1,\dots,n)$. As $X(t)$ take any nonsingular upper triangular matrix-function, continuous at all points except the points t_k, at which $X(t_k-0)$ is the identity matrix and

$$X(t_k+0)=N_{kG}^{-1}(t_k-0)G(t_k+0)N_k^{-1}.$$

Then

$$G(t_k+0)N_k^{-1}X^{-1}(t_k+0)=G(t_k-0)N_k^{-1}X^{-1}(t_k-0),$$

and, consequently, the matrix-function $M(t)=G(t)N^{-1}(t)X^{-1}(t)$ is continuous on Γ. The lemma is proved.

Proof of Theorem 1. Let $\det B(t\pm 0)\neq 0$. The matrix $G=B^{-1}A$ can be represented in the form (3). Clearly, together with the matrix G, the matrix X is also ω-nonsingular; consequently, all diagonal elements of the matrix-function X are ω-nonsingular functions. From (1) it follows that all diagonal elements of the matrix operator $XP+Q$ are Φ-operators in $L_p(\Gamma,\rho)$. Hence it follows immediately that $XP+Q$ are Φ-operators in $L_p^m(\Gamma,\rho)$, and together with it the operator $AP+BQ$ is also a Φ-operator in $L_p^m(\Gamma,\rho)$.

Formula (*) can be derived without difficulty from the equalities (2), (3) and the results of the work ([1]).

We proceed to the proof of the necessity of the conditions of the theorem. We first show that $\det A(t\pm 0)\neq 0$ and $\det B(t\pm 0)\neq 0$. Suppose that the operator $AP+BQ$ is a Φ-operator and that, for at least one of the matrices (for example, for A), this condition is not satisfied. Then it is not difficult to choose a continuous matrix-function $F(t)$ satisfying the following conditions:

1) The norm $\|FP\|$ is so small that the operator $(A+F)P+BQ$ is a Φ-operator.

2) On Γ one can find an arc $t't''(=\gamma)$ such that $A(t)$ is continuous on γ,

$$\det(A+F)(t'+0)\neq 0,\quad \det(A+F)(t''-0)\neq 0,$$

and

$$\det(A+F)(t_0)=0,$$

where t_0 is some point on γ.

Let $Y(t)$ be some continuous matrix-function coinciding with $A+F$ on γ and nonsingular from $\Gamma\setminus\gamma$; then $A+F$ can be represented in the form of a product

$$A+F=ZY,\qquad Z\in\Lambda_m.$$

Since $(A+F)P+BQ$ is a Φ-operator, it follows from the equality

$$(A+F)P+BQ=(ZP+BQ)(PY+Q)+Z(YP-PY)$$

that $PY+Q$ is a Φ_+-operator, and this contradicts the fact that $\det Y(t_0)=0$.

We show that the matrix $B^{-1}A$ is ω-nonsingular. The matrix $B^{-1}A$ can be represented in the form (3). Since $B^{-1}AP+Q$ is a Φ-operator, $XP+Q$ is also a Φ-operator. Hence we already obtain (using the results of the work ([1])) that X (and therefore also $B^{-1}A$) is an ω-nonsingular matrix-function. For the operator K the theorem is proved. It is proved analogously for the operator L as well.

Theorem 2. Let $A(t)$ and $B(t)\in\Lambda_m(\Gamma)$; $t_1,\dots,t_n$ $(\in\Gamma)$ are all discontinuity points of the matrices A and B;

$$\rho=\prod|t-t_k|^{\beta_k}\quad(-1<\beta_k<p-1;\ 1<p<\infty).$$

In order that the operator $K=AP+BQ$ $(L=PA+QB)$ be a Φ-operator in $L_p(\Gamma,\rho)$, it is necessary and sufficient that the following two conditions be fulfilled:

$$\gamma)\quad \det A(t\pm 0)B(t\pm 0)\neq 0\quad\text{for all } t\in\Gamma;$$

$$\delta)\quad \text{for each eigenvalue } \lambda_j^{(k)} \text{ of the matrix}$$

$$A^{-1}(t_k+0)B(t_k+0)B^{-1}(t_k)A(t_k)\quad(B(t_k+0)A^{-1}(t_k+0)A(t_k)B^{-1}(t_k))$$

the relation

$$\beta_k\neq p\alpha_j^{(k)}-1\qquad(j=1,\dots,m;\ k=1,\dots,n)$$

holds, where

$$\alpha_j^{(k)}=(\arg\lambda_j^{(k)})/2\pi\qquad(0\le\alpha_j^{(k)}<1).$$

The formulated theorem obviously follows from Theorem 1 and the following easily proved lemma.

Lemma 3. In order that the matrix $G(t)$ $(\in\Lambda_m(\Gamma))$ be ω-nonsingular, it is necessary and sufficient that the following two conditions be satisfied:

1°. $\det G(t\pm 0)\neq 0$ for all $t\in\Gamma$.

2°. For each eigenvalue $\lambda_j^{(k)}$ of the matrix $G^{-1}(t_k+0)G(t_k)$ the relation

$$\beta_k\neq p\alpha_j^{(k)}-1\quad(j=1,\dots,m;\ k=1,\dots,n),$$

holds, where

$$\alpha_j^{(k)}=(\arg\lambda_j^{(k)})/2\pi\quad(0\le\alpha_j^{(k)}<1).$$

3. As in ([1–3]), the results obtained above can be generalized to the general case when Γ consists of a finite number of closed and open contours.

The results of the present note carry over to the operators K and L acting in certain symmetric spaces (see ([3])). From the theorems of this note one naturally derives necessary and sufficient conditions for the operator corresponding to a system of Wiener–Hopf equations (a discrete analogue) with piecewise-continuous symbol to be a Φ-operator in l_2^m (h_p^m).

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Note: Figure translations are in progress. See original paper for figures.