# ON THE PURITY OF THE SET OF POINTS OF NON-SMOOTHNESS OF A MORPHISM OF SCHEMES

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## Abstract

## Full Text

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*MATHEMATICS*

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# ON THE PURITY OF THE SET OF POINTS OF NON-SMOOTHNESS OF A MORPHISM OF SCHEMES

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**1.** Let $f : X \to Y$ be a locally finitely presented morphism of schemes. Denote by $\mathrm{Sing}^X(f)$ the set of points $x \in X$ at which $f$ is not smooth (recall that one of the equivalent definitions of smoothness of $f$ at a point $x$ consists in saying that $f$ is flat at $x$ and the fiber $f^{-1}(f(x))$ is geometrically regular at $x$). By virtue of ([3]), the set $\mathrm{Sing}^X(f)$ is closed in $X$. Put also $\mathrm{Sing}^Y(f) = f(\mathrm{Sing}^X(f))$. If the morphism $f$ is proper, then $\mathrm{Sing}^Y(f)$ is closed in $Y$.

We say that $f$ satisfies the purity theorem from above if, for every maximal point $x_0$ of the set $\mathrm{Sing}^X(f)$ (i.e. the generic point of an irreducible component of $\mathrm{Sing}^X(f)$), we have

$$\mathrm{codim}(x_0, X) = \dim O_{X,x_0} \leq \dim_{x_0} f + 1.$$

Analogously, if $f$ is proper, we say that $f$ satisfies the purity theorem from below if, for every maximal point $y_0$ of the set $\mathrm{Sing}^Y(f)$, we have

$$\mathrm{codim}(y_0, Y) \leq 1.$$

Below we give several classes of morphisms of schemes satisfying the purity theorem from above (respectively, from below).

**2.** In what follows we shall always assume that all schemes under consideration are locally Noetherian, and that morphisms of schemes are locally of finite type.

If a morphism of schemes $f : X \to Y$ is smooth at a point $x \in X$, then the sheaf of relative differentials $\Omega^1_{X/Y}$ is free at the point $x$, and its rank at $x$ is equal to the relative dimension $\dim_x f$ of the morphism $f$ at the point $x$ ([3]). Let us also note that $f$, moreover, is equidimensional at the point $x$, and consequently, if $X$ and $Y$ are integral, $\dim_x f$ coincides with the dimension of the generic fiber of $f$.

For the proof of the following criterion of smoothness, see ([3]).

Suppose that the sheaf $\Omega^1_{X/Y}$ is free at the point $x$ of rank $\dim_x f$. Then $f$ is smooth at $x$ in any of the following cases:

A. $f$ is flat at the point $x$.

B. $f$ is a morphism of $S$-schemes, $Y$ is smooth over $S$ at the point $f(x)$, and the canonical homomorphism

$$f^*\Omega^1_{Y/S} \to \Omega^1_{X/S}$$

is injective at the point $x$.

**3.** Suppose now that the schemes $X$ and $Y$ are integral. Assume that the morphism $f : X \to Y$ does not satisfy the purity theorem from above. Let $x_0$ be a maximal point of $\mathrm{Sing}^X(f)$ for which

$$\mathrm{codim}(x_0, X) > \dim_x f + 1.$$

Since $x_0$ is maximal on the set $\mathrm{Sing}^X(f)$, $f$ is smooth at every point $x \neq x_0$ such that $x_0 \in \overline{\{x\}}$. Considering the canonical embedding $j : \mathrm{Spec}\, O_{X,x_0} \to X$, we obtain that the restriction of $f$ to $\mathrm{Spec}\, O_{X,x_0}$ is smooth at every point of the punctured local scheme $\overline{X}_{x_0} = \mathrm{Spec}\, O_{X,x_0} - \{x_0\}$. In particular, the sheaf

$$j^*\Omega^1_{X/Y} = \Omega^1_{\overline{X}/Y,x_0}$$

is locally free on the scheme $\overline{X}_{x_0}$. Moreover, if $n$ denotes the dimension of the generic fiber of $f$, then the rank of the sheaf $j^*\Omega^1_{X/Y}$ at every point $x \in \overline{X}_{x_0}$ is equal to $\dim_x f = n$. Since the function $x \rightsquigarrow \dim_x f$ is upper semicontinuous ([3]), we obtain, moreover, that

$$n \leq \dim_x f < \mathrm{codim}(x_0, X) - 1 = \dim O_{X,x_0} - 1.$$

**Proposition.** *Let $A$ be an integral local ring, $\mathfrak{m}$ the maximal ideal of $A$, $X = \mathrm{Spec}\, A$, $X' = X - \{\mathfrak{m}\}$. Let $M$ be a finite $A$-module*

of finite projective dimension type $\leqslant 1$. Suppose that the restriction of the sheaf $L = M^\sim$ to $X'$ is a locally free sheaf of rank $n < \dim A - 1$. Then $M$ is a free $A$-module of rank $n$.

**Proof.** Suppose that $M$ is not free and let

$$0 \to A^l \to A^m \to M \to 0$$

be its projective resolution (free, since $A$ is local). Applying the functor $\mathrm{Hom}(\ , A)$, we obtain the exact sequence

$$A^m \to A^l \to \mathrm{Ext}^1_A(M, A) \to 0.$$

Since $M^\sim$ is locally free on $X$, we have

$$\mathrm{Supp}(\mathrm{Ext}^1(M, A)) = \{\mathfrak{m}\}.$$

By ([1]), Theorem 4, it follows that

$$\dim A \leqslant l - m + 1.$$

On the other hand, obviously $l - m = n$. Thus we obtain $n \geqslant \dim A - 1$, which contradicts the hypothesis of the lemma.

Hence, obviously, it follows

**Theorem 1.** Let $f : X \to Y$ be a morphism of locally finite type of integral locally Noetherian preschemes. Suppose that for every point $x \in X$ we have

$$\dim \operatorname{proj} \Omega^1_{X/Y,x} \leqslant 1$$

and that one of the conditions A or B formulated above is satisfied. Then $f$ satisfies the purity theorem above.

4. **Example 1.** Suppose that $X$ and $Y$ are smooth $S$-preschemes, and $f$ is a morphism of $S$-preschemes smooth at the generic point of $X$ (the latter condition means, of course, that $f$ is dominant and that the corresponding extension $R(X)/R(Y)$ of fields of rational functions is separable). In this case we have an exact sequence of sheaves of relative differentials

$$0 \to f^*\Omega^1_{Y/S} \to \Omega^1_{X/S} \to \Omega^1_{X/Y} \to 0,$$

see ([2]), Lemma 1. This shows that for every point $x \in X$,

$$\dim \operatorname{proj} \Omega^1_{X/Y,x} \leqslant 1,$$

and, moreover, condition B is satisfied. Thus, by Theorem 1, the morphism $f$ satisfies the purity theorem above.

The following example was pointed out to me by V. A. Danilov.

**Example 2.** The hypotheses of Theorem 1 are satisfied if $f : X \to Y$ is a flat morphism of integral schemes which is locally a complete intersection (for example, a quasiprojective flat morphism of regular schemes).

The following result is well known and is a special case of the Zariski–Nagata purity theorem ([4,5]).

**Corollary.** Suppose that $f$ is a quasi-finite morphism of regular schemes. Then $\mathrm{Sing}^X(f)$ coincides with the set of points of $X$ at which $f$ is ramified, and defines a divisor on $X$.

5. Let now $f : X \to Y$ be a proper morphism. Whereas the question of purity above is local in $X$, the question of purity below for the morphism $f$ is local in $Y$, which, obviously, requires consideration of the global properties of the morphism $f$.

In the case where $f$ is a finite morphism of regular preschemes, the assertion that the purity theorem below holds for $f$ is nothing other than the Zariski-Nagata purity theorem. The following result concerns the case of a morphism of relative dimension 1 (i.e. $f$ is equidimensional with one-dimensional fibers).

**Theorem 2.** Let $f : X \to Y$ be a flat proper morphism of relative dimension 1 of integral regular schemes, whose generic fiber is a smooth geometrically connected curve of genus $g$. Suppose, in addition, that in the case $g > 1$, $f$ is cohomologically flat in dimension 0 (i.e. $f_*(O_X) = O_Y$ universally), and in the case $g \geqslant 1$, for all $y \in Y$, $\mathrm{char}(k(y)) = 0$. Then $f$ satisfies the purity theorem below.

We note that in the case when $f$ is separable (i.e., the fibers of $f$ are geometrically reduced), the assertion of the theorem, without any additional conditions on $f$, follows trivially from Theorem 1 (Example 1). In the general case, however, the proof essentially uses the condition that the fibers of $f$ are curves, and requires an interesting study of the properties of singular curves varying in a family.

**Corollary.** *Under the hypotheses of Theorem 2, if $f$ is smooth over an open set $V \subset Y$ containing the points of codimension $\leqslant 1$, then $f$ is smooth on all of $X$.*

6. The restrictions on the cohomological flatness of the morphism $f$ in the case $g > 1$, and on the characteristic in the case $g \geqslant 1$, arise in the course of the proof of Theorem 2 and, as it seems to us, are not essential. More generally, it seems plausible that any flat proper morphism of relative dimension 1 of regular schemes satisfies the purity theorem below. As follows immediately from Theorem 1 (Example 2), this is true, in any case, if $f$ is, in addition, separable and is a locally flat intersection.

In conclusion, I express my gratitude to V. A. Danilov for his constant interest in the work and to M. Raynaud, who pointed out errors in the original proof of Theorem 2.

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*Note: Figure translations are in progress. See original paper for figures.*