ON THE BEHAVIOR OF A GENERALIZED SOLUTION OF A MIXED PROBLEM FOR THE WAVE EQUATION IN A DOMAIN CLOSE TO CLOSED

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Abstract

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ON THE BEHAVIOR OF A GENERALIZED SOLUTION OF A MIXED PROBLEM FOR THE WAVE EQUATION IN A DOMAIN CLOSE TO CLOSED

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Our purpose is to study the behavior of the solution of the mixed problem for the wave equation in the exterior of a domain Ω of "trap" type (see Fig. 1), if the support of the initial data belongs to a set S inside the "trap." The main result of our work is formulated in Theorem 2.

I. Let R_N be N-dimensional Euclidean space ($N \geq 3$), Ω an open bounded set in R_N (not necessarily simply connected), and B an operator given on the boundary of the set $\overline{\Omega}$. By the symbol Ω_1 we denote the largest open connected set contained in $R_N/\overline{\Omega}$ and containing infinitely distant points, and $\Omega_2 = R_N \setminus (\Omega \cup \Omega_1)$. We assume that the domain Ω and the operator B are such that:

1) the boundary of the set $\overline{\Omega}$ belongs to the class $C_{[1,\alpha]}$, $\alpha > 0$*;
2) Ω_2 is either empty, or $\operatorname{mes}\Omega_2 > 0$ and $\rho(\Omega_1, \Omega_2) > 0$;
3) the operator B has the form $(Bu)(x) = u(x) \,|\, x \in \operatorname{bd}\overline{\Omega}$, or

$$(Bu)(x) = [\partial u/\partial n + \sigma(x)u](x) \,|\, x \in \operatorname{bd}\overline{\Omega},$$

where the function $\sigma(x) \geq 0$ is continuous.

Fig. 1

II. Consider the Cauchy problem for the heat equation.

$$\partial u/\partial t = \Delta u, \quad x \in R_N/\overline{\Omega}, \quad t > 0, \quad u(x,t) \in L^\infty; \quad (Bu)u(x,t) = 0,$$

$$u(x,+0) = u_0(x). \tag{1}$$

Let $G(x,y,t)$ be the Green' s function of problem (1). We extend the function $G(x,y,t)$ by putting $G(x,y,t) = 0$, if either $x \in \Omega$, or $y \in \Omega$, and put

$$G_0(x,y,t) = (4\pi t)^{-N/2}\exp(-(x-y)^2/4t); \quad g(x,y,t) = G_0(x,y,t) - G(x,y,t).$$

The operator

$$(G(t)f)(x) = \int G(x, y, t) f(y)\, dy$$

is a bounded operator in any L^p, $1 \le p \le \infty$. On functions from L^p we define the operator

$$Hu = (\text{strong}) \lim_{t \to +0} t^{-1}(E - G(t))u. \tag{2}$$

The operator H, considered in L^2, is self-adjoint, semibounded, and is an extension of the Laplace operator originally given on finite functions in $R_N \setminus \Omega$ satisfying the boundary conditions B.

III. The function $u(x, k)$ is called a solution of the scattering problem for the operator H if $u(x, k) \in L^\infty$ satisfies the equation

$$Hu = \lambda u, \qquad \lambda = k^2, \qquad k \in R_N \tag{3}$$

* For the definition of this class see ([1]).

and is representable in the form

$$u(x, k) = \exp(ikx) + \varphi(x, k), \tag{4}$$

where the function $\varphi(x, k)$ satisfies the radiation conditions

$$\varphi(x, k) = O\left(|x|^{(1-N)/2}\right); \qquad (\partial/\partial|x| - i\sqrt{\lambda})\varphi(x, k) = o\left(|x|^{(1-N)/2}\right), \quad |x| \to \infty. \tag{5}$$

Let

$$T^+(\lambda) = \left[\lim_{\varepsilon \to +0} \left(e^{-(\lambda + i\varepsilon)t} - G_0(t)\right)^{-1}\right] g, \quad t > 0, \quad \lambda > 0.$$

Theorem 1. 1. In order that the function $u(x, k) = \exp(ikx) + \varphi(x, k)$ be a solution of the scattering problem, it is necessary and sufficient that the function $\varphi(x, k)$ satisfy the equation

$$\varphi = T^+(\lambda)(\exp(ikx) + \varphi). \tag{6}$$

2. Either the operator

$$(E - T^+(\lambda))^{-1} \in [L^p \to L^p,\ 2N/(N-1) < p < \infty]$$

exists and equation (6) has a unique solution, or in L^p, $2N/(N-1) < p < \infty$, there exists a nontrivial solution of the equation

$$T^{+}(\lambda)\psi = \psi, \tag{7}$$

and the function $\psi(x, \lambda)$ is a solution of equation (7) if and only if it satisfies the equation

$$H\psi = \lambda\psi.$$

3. Those numbers λ for which there exists a nontrivial solution of equation (7) are eigenvalues of the discrete spectrum of the operator H and form at most a countable set $\{\lambda_i\}$, while the functions $\psi(x, \lambda_i)$ satisfying (7) belong to L^∞ and are nonzero only when $x \in \Omega_2$. (It follows from this that if Ω_2 is empty, then $\{\lambda_i\}$ is empty as well.)

With the aid of the functions $u(x, k)$ and $\psi(x, \lambda_i)$ one constructs the spectral resolution of the operator H. The spectral function $E(\lambda, H)$ of the operator H is an integral operator with kernel

$$\theta(\lambda, x, y) = (2\pi)^{-N} \int_{|k|^2<\lambda} u^*(x, k) u(y, k)\, dk + \sum_{\lambda_i<\lambda} \psi(x, \lambda_i)\psi(y, \lambda_i).$$

IV. Definition. The function $v(x, t)$ is called a **generalized solution of the mixed problem for the wave equation** if it is twice differentiable with respect to t, for $t > 0$, strongly in the sense of the metric of L^2, satisfies the equation

$$\partial^2 v/\partial t^2 = Hv \tag{8}$$

and the initial conditions

$$\lim_{t\to+0} \|v(\,\cdot\,, t) - \beta\|_2 = 0, \qquad \lim_{t\to+0} \|v_t(\,\cdot\,, t) - \alpha\|_2 = 0. \tag{9}$$

Lemma 1. If

$$\langle\alpha, \alpha\rangle + \langle\alpha, H\alpha\rangle - \langle\beta, \beta\rangle + \langle H\beta, H\beta\rangle = C_0 < \infty,$$

then the function

$$v(\,\cdot\,, t) = \int_0^\infty \cos\sqrt{\lambda}\, t\, d_\lambda E(\lambda, H)\beta + \int_0^\infty \frac{\sin\sqrt{\lambda}\, t}{\sqrt{\lambda}}\, d_\lambda E(\lambda, H)\alpha$$

is a generalized solution of problem (8)–(9).

V. Let $\Omega^{(\tau)}$ and $B^{(\tau)}$, $0 \le \tau \le 1$, be a family of domains and boundary operators depending on the parameter τ and for each τ satisfying

conditions 1)–3). Let $G(\tau \mid x, y, t)$ be the corresponding Green functions of problem (1), and let $H(\tau)$ be the operators defined by formula (2). Suppose that the domain Ω and the operator B are such that, for some $t = t_0 > 0$, the following additional conditions are satisfied:

4) There exist constants C_1, C_2, and C_3, independent of τ, such that for all $\tau \in [0,1]$ the estimates

$$|g(\tau \mid x, y, t_0)| \le C_1 \exp\left(-\frac{1}{8t_0}(x-y)^2\right), \qquad x, y \in R_N;$$

$$\int |g(\tau \mid x, y, t_0)|\, dy \le C_2 \exp\left(-\frac{1}{8t_0}|x|^2\right), \qquad x \in R_N;$$

$$\int |\nabla_x g(\tau \mid x, y, t_0)|\, dy \le C_3 \exp\left(-\frac{1}{8t_0}|x|^2\right), \qquad |x| \ge 2R_0,$$

hold, where R_0 is the radius of a ball containing any of the sets $\Omega^{(\tau)}$, $0 \le \tau \le 1$.

5)

$$\lim_{|\tau_1-\tau_2|\to 0} \int_\Omega |g(\tau_1 \mid x, y, t_0) - g(\tau_2 \mid x, y, t_0)|\, dx\, dy = 0, \qquad \tau_1, \tau_2 \in [0,1].$$

We give an example of a family satisfying conditions 1)–5).

$$\Omega^{(\tau)} = \{x = (r\sin\theta\sin\varphi,\ r\sin\theta\cos\varphi,\ r\cos\theta);$$

$$R_1 < r < R_2, \quad 0 \le \varphi \le 2\pi, \quad (1-\tau)\pi/2 < \theta \le \pi\}.$$

$$(B^{(\tau)}u)(x) = u(x) \mid x \in \mathrm{gr}\,\overline{\Omega}^{(\tau)}.$$

For simplicity we shall assume, in addition, that for any $\tau \in [0,1)$ the domain $\Omega^{(\tau)}$ is simply connected, while for $\tau = 1$ it is not simply connected.

Let $\{\lambda_i\}$ be the eigenvalues of the discrete spectrum of the operator $H(1)$. If conditions 1)–5) are satisfied, then the following is valid.

Theorem 2. Let $v(\tau \mid x, t)$ be the solution of problem (8)–(9),

$$S \subset \Omega_2^{(1)}; \quad \inf_{0 \le \tau \le 1} \rho(S, \Omega^{(\tau)}) > 0; \quad \alpha(x) \in W_2^1; \quad \beta(x) \in W_2^2;$$

the support of the functions $\alpha(x)$ and $\beta(x)$ is contained in the set S, ε is an arbitrary positive number, and

$$\lambda(\varepsilon) > 8\varepsilon^{-2}[\|\alpha\|_{W_2^1} + \|\beta\|_{W_2^2}].$$

Then there exist a $\delta < 1$ and a function $\sigma(\tau) \to 0$ as $\tau \to 1$, such that for all $\tau \in [\delta, 1)$, $t > 0$, the inequality

$$\left\| v(\tau\,|, t) - \sum_{\lambda_j < \lambda(\varepsilon)} \left[\int_{\lambda_j - \sigma(\tau) < \lambda < \lambda_j + \sigma(\tau)} \cos\sqrt{\lambda}\, t \, d_\lambda E(\lambda, H(\tau))\beta + \right.\right.$$

$$\left.\left. + \int_{\lambda_j - \sigma(\tau) < \lambda < \lambda_j + \sigma(\tau)} \frac{\sin\sqrt{\lambda}\, t}{\sqrt{\lambda}} \, d_\lambda E(\lambda, H(\tau))\alpha \right] \right\|_2 < \varepsilon$$

is satisfied.

The proof of Theorem 2 is based on Theorem 1 and Lemma 2.

Lemma 2. The operator $T^+(\tau \mid \lambda)$ is continuous in the uniform operator topology of the space

$$[L^p \to L^p, \ 2N/(N-1) < p < \infty]$$

with respect to the aggregate of variables $\tau \in [0, 1]$, $\lambda \in [0, \infty)$.

It follows from Theorem 2 that if the domain $\Omega^{(\tau)}$ is sufficiently close to a closed one (for this it is sufficient to require that the integral

$$\int |g(1 \mid x, y, t_0) - g(\tau \mid x, y, t_0)| \, dx \, dy$$

be sufficiently small), then in the continuous spectrum of the operator $H(\tau)$ only intervals in neighborhoods of the discrete eigenvalues of the closed domain are essential.

Lemma 3. If λ_j is a simple eigenvalue of the discrete spectrum of the operator $H(1)$, then the point $\mu = 1$ is a pole of first order for the function

$$(\mu E - T^+(1 \mid \lambda_j))^{-1}.$$

* We note that A. A. Samarskii had already proved earlier (2) that a perturbation of the spectrum of the Helmholtz operator depends on the change in the capacity of the boundary.

It is easy to see that the function $(\mu E - T^{+}(1|\lambda_j) - \Delta T)^{-1}$, for $\Delta T = T^{+}(\tau|\lambda) - T^{+}(1|\lambda_j)$ sufficiently small in a neighborhood of the point $\mu = 1$, has a pole of first order. Expanding the function $(\mu E - T^{+}(1|\lambda_j) - \Delta T)^{-1}$ in powers of ΔT in a neighborhood of this pole, it is easy to obtain the asymptotics of the function $u(\tau|x, k)$ as $\tau \to 0$ and $|k^2 - \lambda_j| \to 0$.

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Note: Figure translations are in progress. See original paper for figures.