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Abstract

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MATHEMATICS

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ON THE PROBLEM OF INVARIANCE OF DYNAMIC SYSTEMS

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The problem of invariance of dynamic systems consists in determining conditions under which certain quality indices of a system do not depend on external disturbances acting upon it. Works by G. V. Shchipanov, N. N. Luzin, V. S. Kulebakin, B. N. Petrov, and others (see ([1])) have been devoted to the problem of invariance and to the development of methods for constructing disturbance-free technical devices. L. I. Rozonoer ([2]), in studying the problem of invariance, applied a variational approach using the apparatus of the theory of optimal systems ([3,4]). In the present work the variational approach is developed in the direction of investigating large variations of the functional. This makes it possible to obtain necessary and at the same time sufficient conditions for invariance of the terminal state of a nonlinear dynamic system, revealing a deep analogy with the corresponding optimality conditions ([3–6]), and on this basis to solve the problem of synthesizing a regulator ensuring invariance of a given nonlinear object.

1°. Statement of the problem. Let the motion of a disturbed dynamic system be described by an n-dimensional system of equations

$$\dot{x}(t) = f(x, u, t), \tag{1}$$

where $x(t)$ is the vector of phase coordinates and $u(t)$ is the vector of external disturbing actions. Consider the functional

$$I(x, u) = \Phi[x(T), T]. \tag{2}$$

The instant T is determined from the condition that the right end $\{x(T), T\}$ of the trajectory of system (1) reaches the hypersurface M, given by the equation

$$M(x, t) = 0. \tag{3}$$

Let a domain $A \subset X \times t$ be given. We shall say that system (1) is Φ-invariant on M with respect to u in A if, for any of its trajectories $x(t), t$, beginning at a point $\{y, \tau\} \in A$, the value of functional (2) does not depend on the disturbing vector $u(t)$. We pose the problem of determining the conditions ensuring invariance of system (1) in the indicated sense. The formulation considered is a generalization of the problem of weak invariance ([2]).

We shall assume the following assumptions to be satisfied: $u(t) \in R(y, \tau)$, where $R(y, \tau)$ is the set of all piecewise-continuous disturbances under which the trajectory of system (1), beginning at the point $\{y, \tau\}$, does not leave the boundaries of the closed domain $G \supset A$ of definition of the right-hand sides of system (1) and reaches M in a finite interval of time; there exists at least one vector-function $\tilde{u}(t) \in R(y, \tau)$ for all $\{y, \tau\} \in A$; the functions $\Phi(x, t)$ and $M(x, t)$ in G are independent, continuous in their arguments together with their first partial derivatives, and have bounded second partial derivatives; $f(x, \tilde{u}(t), t)$ possesses these properties with respect to x and is continuous in t; $f(x, u, t)$ is piecewise-continuous in the totality of its arguments (discontinuities of the first-

of this kind on smooth hypersurfaces in the space $X \times U \times t$); the corresponding $\tilde{u}(t)$-trajectories of system (1) do not touch M.

2°. **Exact formula for the change of the functional.** Construct in A the field of trajectories $\tilde{x}(t), t$ of system (1) corresponding to one and the same **supporting perturbation**–the fixed vector-function $u(t) = \tilde{u}(t)$. Through each point $\{y, \tau\} \in A$ there passes a unique trajectory $\tilde{x}(t), t$, and, if A is a domain of invariance, every trajectory $\tilde{x}(t), t$ on the interval $[\tau, \tilde{T}]$, where $\tilde{T}$ is the time of reaching M, is entirely contained in A.

Define along each trajectory $\tilde{x}(t), t$ the vector-function $\tilde{p}(t)$ by the equation

$$\dot{\tilde{p}}(t) = -\operatorname{grad}_x H(\tilde{x}, \tilde{p}, \tilde{u}, t), \qquad H(\tilde{x}, \tilde{p}, \tilde{u}, t) \equiv (\tilde{p}, f(\tilde{x}, \tilde{u}, t)) \tag{4}$$

with the boundary condition

$$\tilde{p}(\tilde{T}) = \left[-\operatorname{grad}_x \Phi(x, t) + \left(\frac{d\Phi(x, t)}{dt} \middle/ \frac{dM(x, t)}{dt}\right) \operatorname{grad}_x M(x, t)\right]_{t=\tilde{T},\ x=\tilde{x}(\tilde{T})}.$$

Equation (4) assigns to each point $\{y, \tau\} \in A$ the vector $\tilde{p} = \tilde{p}(y, \tau)$. We shall call the vector field $\tilde{p}(y, \tau)$ the **supporting field**.

The trajectory $\tilde{x}(t), t$ passing through each point $\{y, \tau\} \in A$ assigns to it, in a one-to-one manner, the value of the functional (2), which thus turns out to be a function of the point $\{y, \tau\}$. We shall call this function

$$\tilde{V}(y, \tau) \equiv \Phi[\tilde{x}(\tilde{T}), \tilde{T}] \tag{5}$$

the **supporting function**.

Compute the difference

$$\Delta I = I(\hat{x}, \hat{u}) - I(\tilde{x}, \tilde{u}) = \Phi[\hat{x}(\hat{T}), \hat{T}] - \Phi[\tilde{x}(\tilde{T}), \tilde{T}]$$

between the values of the functional (2) for trajectories $x(t), t$ and $\tilde{x}(t), t$ of system (1) passing through one and the same point $\{x^0, t^0\} \in A$, corresponding to the vector-functions $u(t)$ and $\tilde{u}(t)$. We have

$$\Delta I = \tilde{V}[\hat{x}(\hat{T}), \hat{T}] - \tilde{V}(x^0, t^0).$$

At the same time, for the derivative of the supporting function $\tilde{V}(x,t)$, corresponding to $\tilde{u}(t)$, along the trajectory $\hat{x}(t), t$, from the approximate formula for small changes of the functional (7) we have

$$d\tilde{V}(x,t)/dt = -(\tilde{p}(x,t), [f(x, \hat{u}, t) - f(x, \tilde{u}, t)]).$$

Hence

$$\Delta I = -\int_{t^0}^{\hat{T}} [H(\hat{x}, \tilde{p}(\hat{x}, t), \hat{u}, t) - H(\hat{x}, \tilde{p}(\hat{x}, t), \tilde{u}, t)]\, dt. \tag{6}$$

Formula (6) remains valid if $f(x, \tilde{u}(t), t)$ is discontinuous in t, and can be used for constructing algorithms for the numerical solution of optimal-control problems.

An approach using the study of large changes of the functional proves useful in the investigation of sufficient conditions of optimality ([6]).

3°. **Conditions of invariance with respect to a perturbation.** Directly from formula (6) we obtain the following assertion, which, in order to emphasize its analogy with the optimality conditions in the form of the maximum principle ([3–6]), may be called the **principle of independence**.

Theorem 1. *In order that system (1) be Φ-invariant in A on M with respect to u, it is necessary and sufficient that in A, on the supporting field corresponding*

corresponding to some reference disturbance $\tilde{u}(t)$, the function

$$\tilde{H}(x, u, t) \equiv (\tilde{p}(x,t), f(x, u, t))$$

does not depend explicitly on u.

The functions $\tilde{V}(y, \tau)$, $\tilde{p}(y, \tau)$ and $\tilde{H}(y, \tau) \equiv H(y, \tilde{p}(y, \tau), \tilde{u}(\tau), \tau)$ are related by

$$\tilde{p}(y, \tau) = -\operatorname{grad}_y \tilde{V}(y, \tau), \qquad \tilde{H}(y, \tau) = \partial \tilde{V}(y, \tau)/\partial \tau. \tag{7}$$

Therefore, together with the reference function, the reference field and the Hamiltonian of the invariant system are invariant with respect to $\tilde{u}(t)$.

4°. Structure of the invariance domain and invariance with respect to initial data. Under the adopted assumptions the function $\tilde{V}(y,\tau)$ is continuously differentiable with respect to its arguments, and the vector $\operatorname{grad}_{y,\tau}\tilde{V}(y,\tau) = \{-\tilde{p}(y,\tau), \tilde{H}(y,\tau)\} \equiv \tilde{P}(y,\tau)$ does not vanish identically in any subdomain of the domain A. On the other hand, along a trajectory of the invariant system, for any $u(t) \in R(x,t)$, $d\tilde{V}(x,t)/dt \equiv 0$. It follows that the invariance domain A decomposes into smooth manifolds $\tilde{V}_c = \{\{y,\tau\} \in A \mid \tilde{V}(y,\tau) = c\}$ of dimension $l \leq n$, each of which, together with any of its points, contains all trajectories of system (1) issuing from that point. Thus, the invariant system is not completely controllable.

Let a set $L \subset A$ be given. We shall call system (1) **invariant with respect to initial data on** L if the values of functional (2) do not depend on the disturbance $u(t)$ and coincide for all trajectories of system (1) beginning in L. From the structure of A it follows that, for invariance of system (1) with respect to initial data on L, it is necessary and sufficient that the condition of Theorem 1 be satisfied and that $L \subset \tilde{V}_c$. Let L be a smooth manifold and let $l(x,t)$ be an arbitrary tangent vector to L at the point $\{x,t\}$.

Theorem 2. *For system (1) to be invariant with respect to initial data on L, it is necessary and sufficient that the condition of Theorem 1 be satisfied and, at each point $\{x,t\} \in L$,*

$$(\tilde{P}(x,t), l(x,t)) = 0.$$

5°. Synthesis of invariant systems. Suppose there is a system

$$\dot{x}(t) = f(x,u,v,t), \tag{8}$$

where v is a scalar parameter characterizing its variable part. We pose the problem of finding a **correcting function**

$$v = v(x,u,t), \tag{9}$$

for which the system

$$\dot{x}(t) = f[x,u,v(x,u,t),t]$$

is Φ-invariant in A on M with respect to u. Suppose that for $u(t) \equiv 0$ system (8) functions in the desired manner when $v \equiv 0$, and that the vector functions $f(x,0,0,t)$ and $f(x,u,v,t)$ possess the properties indicated in 1° for $f(x,\tilde{u},t)$

and $f(x,u,t)$. We shall assume the function $v(x,u,t)$ to be piecewise continuous in the totality of its arguments.

Construct the reference field $\tilde{p}(y,\tau)$ for system (8), corresponding to $u \equiv 0$, $v \equiv 0$, and write the equation

$$(\tilde{p}(x,t), f(x,u,v,t)) = (\tilde{p}(x,t), f(x,0,0,t)), \tag{10}$$

which determines v as a function of x, u, and t. From Theorem 1 it follows that, in order for function (9) to be a correcting function, it is necessary and sufficient that it satisfy equation (10) in A.

The use of equation (10) makes it possible to solve the problem of synthesizing the correcting function on the basis of predicting the unperturbed motion of the system. This equation can be written in explicit form if

the complete integral of system (8) is known for $u \equiv 0$, $v \equiv 0$. Indeed, let this integral be written in the form $\tilde{x}(t) = \varphi(y,\tau,t)$. Then, using condition (3), we find $\hat{T} = \tilde{T}(y,\tau)$ and $\tilde{x}(\tilde{T}) = \varphi(y,\tau,\tilde{T}) = \psi(y,\tau)$. Hence, by virtue of (5), $\tilde{V}(y,\tau) = \Phi[\psi(y,\tau), \tilde{T}(y,\tau)]$, and the vector-function $\tilde{p}(y,\tau)$ is determined by equality (7).

The results presented admit a generalization to the case of invariance of system (1) simultaneously with respect to $m \geqslant 2$ functionals of the form (2). In solving the problem of synthesis, the control parameter v must in this case be an m-dimensional vector, which makes it possible to construct m correcting functions. If the vector-function $f(x,\tilde{u}(t),t)$ is discontinuous, the results set forth remain valid if the equation (4) defining the vector-function $\tilde{p}(y,\tau)$ is supplemented by the usual jump conditions (7).

6°. Example. In the problem

$$\dot{x}_1 = x_2[1 + u(1 - v^2)], \qquad \dot{x}_2 = x_1 + v(x_1^2 - u^2);$$

$$\Phi(x_1,x_2,t) = x_1^2 + x_2^2; \qquad M(x_1,x_2,t) = x_1 + x_2 - 1$$

we have

$$\tilde{V} = 0.5[1 + (x_1^2 - x_2^2)^2], \qquad \tilde{p}_1 = -2x_1(x_1^2 - x_2^2), \qquad \tilde{p}_2 = 2x_2(x_1^2 - x_2^2),$$

and the correcting functions are determined from the equation

$$v^2 x_1 u + v(x_1^2 - u^2) - x_1 u = 0.$$

Imposing on v the additional condition of minimality in absolute value, we have

$$v = x_1^{-1}u \quad \text{for } |x_1| > |u|, \qquad v = -x_1 u^{-1} \quad \text{for } |x_1| \leqslant |u| \text{ and } u \neq 0,$$

$$v = 0 \quad \text{for } u = 0.$$

A correcting function continuous in the entire space $X \times U \times t$ does not exist in this problem.

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Note: Figure translations are in progress. See original paper for figures.