ON SUBSPACES OF DYADIC BICOMPACTS

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MATHEMATICS

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ON SUBSPACES OF DYADIC BICOMPACTS

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The depth gX of a topological space X will mean the supremum of the cardinalities of closures of discrete subspaces $T \subset X$. A. V. Arhangel' skii (1) posed the question: for what spaces is $gX = |X|$?* It is easy to construct examples of Hausdorff spaces X for which $gX < |X|$. Here we shall give a scheme for proving that, under certain assumptions of set theory, the depth of every dyadic bicompact R is equal to its cardinality. Moreover, we shall show that gR is always attained, i.e. there exists a discrete $T \subset R$ such that $|\overline{T}| = |R|$. It is further shown that if $(wR)^{\aleph_0} = wR$, then R contains all extremally disconnected spaces of weight $\leq wR$. These results follow from the following main theorem.

Main theorem. *Let R be a dyadic bicompact of weight τ, and suppose that one of the following conditions is satisfied:* 1) $\tau = \aleph_{\alpha+1}$ *or* 2) $\aleph_1 \leq cf(\tau) < \tau$. *Then R contains a bicompact $X \subset R$ which is continuously mapped onto D^τ.*

Here $cf(\tau)$ denotes the cofinal character of the number τ, the minimum of the cardinalities of sets A such that

$$\tau = \sum_{\alpha \in A} \mathfrak{n}_\alpha$$

and $\mathfrak{n}_\alpha < \tau$. The remaining terms and notation are taken from the author' s previous works (2–4).

§ 1. A system $\mathfrak{B} = \{U_\alpha\}$ consisting of open subsets of a space X will be called a local π-base of a set $F \subset X$ if for every neighborhood OF there exists a $U_\alpha \in \mathfrak{B}$ such that $U_\alpha \subset OF$. The minimum of the cardinalities of local π-bases of F will be called the π-character of F and denoted by $\pi\chi(F, X)$. If F is a point $x \in X$, then $\pi\chi(x, X)$ will be called the π-character of the point $x \in X$. We note that the local characteristic of a space "π-character"is analogous to the integral weight characteristic "π-weight," introduced by V. I. Ponomarev (5). Let $f : Y \to X$ be a continuous mapping of the space Y onto X, and let $F \subset X$. Then by the ν-character of the set F in X relative to Y and f (notation $\nu(F, X, Y, f)$) we shall mean the minimum of the neighborhood characters $\chi(\Phi, Y)$ of sets $\Phi \subset f^{-1}F$. In what follows we shall assume everywhere that $X = R = f(D^\tau)$ is a dyadic bicompact. Therefore $Y = D^\tau$, f is some mapping of D^τ onto R, and $\nu(F, X, Y, f) = \nu(F, R, D^\tau, f) = \nu(F, R)$.

Lemma 1. *Let $f : D^\tau \to R$ be some mapping of D^τ onto R, and let F be a closed nowhere dense subset of R, with $\pi\chi(F, R) \leq \mathfrak{n}$ and $wF \leq \mathfrak{n}$. Then there exists a point $x \in F$ such that $\nu(x, R) \leq \mathfrak{n}$.*

Corollary 1. *For any point of a dyadic bicompact R the inequality holds*

$$\nu(x, R) \leq \pi\chi(x, R) \leq \delta(x, R) \leq \chi(x, R) \leq wR, \tag{1}$$

where $\delta(x, R)$ is the δ-character of the point x in R (see ([3,4])*).*

Theorem 1. *Let R be a dyadic bicompact and M an arbitrary subspace of R. If $\nu(x, R) \leq \mathfrak{m} \geq \aleph_0$ for all $x \in M$, then $w\overline{M} \leq \mathfrak{m}$.*

The proof of this theorem is completely analogous to the proof of Theorem 14 from ([2]). We note that, by virtue of inequality (1), Theorem 1 remains valid if in its formulation the ν-character is replaced by the $\pi\chi$-, δ-, or χ-character.

Corollary 2. Let

$$E(\mathfrak{n}) = \{x \in R,\ \nu(x, R) \leq \mathfrak{n}\}, \qquad F(\mathfrak{n}) = \overline{E(\mathfrak{n})}.$$

If R is dyadic and $\mathfrak{n} < wR$, then

$$U = R \setminus F(\mathfrak{n}) \neq \emptyset.$$

§ 2. Preimages of D^τ embedded in dyadic bicompacts

Theorem 2. Let R be a dyadic bicompactum and let $\nu(x, R) \geq \mathfrak{n} \geq \aleph_0$ for all points $x \in R$, and let there be a mapping $f : D^\tau \to R$. Then there exists a closed subset $F \subset R$ which is continuously mapped onto $D^\mathfrak{n}$.

Proof. Denote by

$$I^\mathfrak{n} = \prod_{\alpha \in B} I_\alpha$$

the Tikhonov product of the intervals $I_\alpha = \{0 \leq x \leq 1\}$. The generalized Cantor discontinuum

$$D^\mathfrak{n} = \prod_{\alpha \in B} D_\alpha$$

may be regarded in a natural way as embedded in $I^\mathfrak{n}$, if one assumes that $D_\alpha = \{0, 1\} \subset I_\alpha$. We shall construct a continuous mapping $\varphi : R \to I^\mathfrak{n}$ such that $\varphi(R) \supset D^\mathfrak{n}$. Then $F = \varphi^{-1}(D^\mathfrak{n})$ will be the required set. We define

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* $|X|$ is the cardinality of the set X; wX is the weight of the space X.

the mapping φ by a system of mappings $\{\varphi_\alpha\}$, with each $\varphi_\alpha : R \to I_\alpha$ and $\varphi_\alpha(R) \supset D_\alpha$. We shall construct this system by transfinite induction. Namely, we construct a system of pairs

$$\omega_\alpha = \{A_\alpha^0, A_\alpha^1\}, \qquad \alpha \in B,$$

of the bicompactum R, consisting of disjoint closed sets of type G_δ in R such that $|B| = \mathfrak{n}$ and for any finite set of indices $\alpha_1, \dots, \alpha_s$, $\alpha_i \in B$, and the corresponding set of zeros and ones $i_1, \dots, i_s$, we have

$$A_{\alpha_1}^{i_1} \cap \dots \cap A_{\alpha_s}^{i_s} \neq \emptyset.$$

Then, for each pair ω_α, define a continuous real-valued function $\varphi_\alpha : R \to I_\alpha$ such that

$$A_\alpha^0 = \varphi^{-1}(0), \qquad A_\alpha^1 = \varphi^{-1}(1).$$

The system of functions $\{\varphi_\alpha\}$ determines in the natural way a mapping

$$\varphi = \prod_{\alpha \in B} \varphi_\alpha$$

of the bicompactum R into $I^{\mathfrak{n}}$ by the rule

$$y = \varphi(x) = \{\varphi_\alpha(x)\} \in I^{\mathfrak{n}}.$$

Let

$$x = \{i_\alpha\} \in D^{\mathfrak{n}}, \qquad \alpha \in B;$$

then

$$\varphi^{-1}(x) = \bigcap_{\alpha \in B} A_\alpha^{i_\alpha}.$$

But the latter intersection is nonempty in R, since the family

$$\{A_\alpha^{i_\alpha}\}, \qquad \alpha \in B,$$

is centered (the α' s are distinct!). Thus $\varphi(R) \supset D^{\mathfrak{n}}$. Let us construct the required system of pairs $\omega_\alpha = \{A_\alpha^0, A_\alpha^1\}$. For $\omega_1 = \{A_1^0, A_1^1\}$ take an arbitrary pair consisting of disjoint closed sets of type G_δ in R. Suppose that for all ordinal numbers α less than some $\beta < \Omega(\mathfrak{n})$, pairs

$$(A_1^0, A_1^1), \dots, (A_\alpha^0, A_\alpha^1) \dots, \qquad \alpha < \beta,$$

have been constructed such that: 1) $A_\alpha^0 \cap A_\alpha^1 = \emptyset$; 2) A_α^i are closed of type G_δ in R; 3) for any set $\alpha_1, \dots, \alpha_s$, $\alpha_i < \beta$, and any set $i_1, \dots, i_s$ of zeros and ones, we have

$$A_{\alpha_1}^{i_1} \cap \dots \cap A_{\alpha_s}^{i_s} \neq \emptyset.$$

Consider the mapping

$$\psi = \prod_{\alpha < \beta} \varphi_\alpha, \qquad \psi : R \to I^{|\beta|},$$

where $|\beta| = |\{\alpha, \alpha < \beta\}|$. As we showed earlier, $\psi(R) \supset D^{|\beta|}$. Moreover, since $f_\alpha(x) = i$ $(i = 0, 1)$ if and only if $x \in A^i_\alpha$, we have

$$\psi^{-1}(x) = \bigcap_{\alpha<\beta} A^{i_\alpha}_\alpha, \qquad \text{if } x = \{i_\alpha\} \in D^{|\beta|}.$$

Put $F_x = \psi^{-1}(x)$. Since

$$wI^{|\beta|} = |\beta|,$$

there exists a bicompactum $Y \subset R$ such that

$$\psi(Y) = \psi(R)$$

and

$$wY \leq |\beta|$$

(see (6)). Thus

$$Y \cap F_x \neq \emptyset$$

for all $x \in D^{|\beta|}$. We shall show that there exists a neighborhood OY of the bicompactum Y in R such that

$$(R \setminus OY) \cap F_x \neq \emptyset$$

for all $x \in D^{|\beta|}$. Suppose the contrary. This means that for every neighborhood OY there exists $x \in D^{|\beta|}$ such that $F_x \subset OY$. By the bicompactness of R and the continuity of ψ, there exists a basic open set $U \subset \psi(R)$ such that

$$x \in U$$

and

$$\psi^{-1}U \subset OY.$$

Since

$$w(\psi R) \leq wI^{|\beta|} = |\beta|,$$

it follows from this that

$$\pi\chi(Y, R) \leq |\beta|.$$

On the other hand,

$$wY \leq |\beta|,$$

and hence, by Lemma 1, we obtain that there exists a point $y \in Y$ for which

$$\nu(y, R) \leq |\beta| < \mathfrak{n},$$

which contradicts the assumption. (We note that Y is nowhere

not dense, for otherwise for any point $y \in \operatorname{int} Y \neq \emptyset$ we would have $\nu(x, R) \leq uY \leq |\beta| < \mathfrak{n}$. Thus there exists a neighborhood OY such that $(R \setminus OY) \cap F_x \neq \emptyset$ for all $x \in D^{|\beta|}$. Put $Z = R \setminus OY$. Then Y and Z are disjoint closed sets, and, by normality of R, there exists a real function φ_β such that $\varphi_\beta^{-1}(0) \supset Z$,

$\varphi_\beta^{-1}(1) \supset Y$, and $\varphi_\beta^{-1}(0) \cap \varphi_\beta^{-1}(1) = \emptyset$. Put $A_\beta^0 = \varphi_\beta^{-1}(0)$, $A_\beta^1 = \varphi_\beta^{-1}(1)$. The pair $\omega_\beta = (A_\beta^0, A_\beta^1)$ is the desired one. Indeed, the centeredness of the system $\{A_\alpha^{i_\alpha}\}$, $\alpha \leq \beta$, for distinct α, follows from the fact that

$$Y \cap \bigcap A_\alpha^{\alpha_i} \neq \emptyset \quad \text{and} \quad Z \cap \bigcap_{\alpha<\beta} A_\alpha^{i_\alpha} \neq \emptyset.$$

Theorem 2 is completely proved.

Corollary 3. *If a dyadic bicompactum R contains a set F of type G_δ (in particular, an open one), and moreover $\nu(x, R) \geq \mathfrak{n} \geq \aleph_0$ for all $x \in F$, then F (and, consequently, R) contains a closed X which is continuously mapped onto $D^\mathfrak{n}$.*

§ 3. **Proof of the main theorem.** 1) $\tau = \aleph_{\alpha+1}$. Put $\mathfrak{n} = \aleph_\alpha$; then, by Corollary 2, there exists a nonempty open set $U = R \setminus F(\mathfrak{n}) \subset R$ such that $\nu(x, R) = \nu(x, U) \geq \tau$. Hence, by Corollary 3, there exists a closed $X \subset R$ continuously mapped onto D^τ.

2) $\aleph_1 \leq cf(\tau) < \tau$. In this case

$$\tau = \sum_{\alpha \in A} \mathfrak{n}_\alpha, \quad \mathfrak{n}_\alpha < \tau \quad \text{and} \quad |A| = cf(\tau) = \mathfrak{m} < \tau.$$

Let $F_\alpha = F(\mathfrak{n}_\alpha)$ be the sets defined in Corollary 2. Without loss of generality we shall assume that the sequence

$$F_1 \subset F_2 \subset \cdots \subset F_\alpha \subset \cdots, \quad \alpha < \Omega(\mathfrak{m}),$$

is strictly increasing, well ordered by the uncountable regular ordinal $\Omega(\mathfrak{m})$. Then

$$\Phi_1 \subset \Phi_2 \subset \cdots \subset \Phi_\alpha \subset \cdots, \quad \alpha < \Omega(\mathfrak{m}),$$

where $\Phi_\alpha = f^{-1}F_\alpha$ and $f : D^\tau \to R$ is some mapping of D^τ onto R, is an analogous sequence in D^τ. By a theorem of H. Shanin on calibers (^{7}),

$$L = \bigcap_{\alpha \in A} (D^\tau \setminus \Phi_\alpha) \neq \emptyset,$$

if $A = \{\alpha, \alpha < \Omega(\mathfrak{m})\}$.

Let $x \in L$. For each $\alpha \in A$ choose a basic neighborhood

$$U_\alpha(x) = H_{\alpha_1 \dots \alpha_s}^{i_1 \dots i_s}$$

of the point x in D^τ (^{2}) such that $U_\alpha(x) \subset D^\tau \setminus \Phi_\alpha$. Then $\bigcap_{\alpha\in A} U_\alpha(x) \subset L$, but the intersection

$$\bigcap_{\alpha\in A} U_\alpha(x) = H_w^{i(w)}$$

is a layer of D^τ (see (^{2})), and moreover $|w| = |A| = \mathfrak{m}$; consequently, $H_w^{i(w)}$ is homeomorphic to D^τ (^{2}), since $\mathfrak{m} < \tau$. Thus,

$$H_w^{i(w)} \subset L \quad \text{and} \quad f\Big(H_w^{i(w)}\Big) \subset \bigcap_{\alpha\in A} (R \setminus F_\alpha).$$

Note that

$$f\Big(H_w^{i(w)}\Big) = X_0$$

is a dyadic bicompactum, and for every point $x \in X_0$ we have $\nu(x, R) = \tau$. We shall prove that in fact $\nu(x, X_0) = \tau$, if

$$f : H_w^{i(w)} \to X_0$$

is the restriction of the mapping f to $H_w^{i(w)}$. Indeed, if there existed a point $x_0 \in X_0$ such that $\nu(x_0, X_0) = \mathfrak{n} < \tau$, then there would exist a set $Z \subset \tilde{f}^{-1}(x_0)$ and $\chi(Z, H_w^{i(w)}) = \mathfrak{n}$. Then

$$\chi(Z, D^\tau) \leq \chi(Z, H_w^{i(w)}) \cdot \chi(H_w^{i(w)}, D^\tau) = \mathfrak{n} \cdot |w| = \mathfrak{n} \cdot \mathfrak{m} < \tau.$$

Thus $\nu(x_0, R) = \mathfrak{n} \cdot \mathfrak{m} < \tau$, contrary to what was proved earlier. Hence there exists a dyadic bicompactum $X_0 \subset X$ such that $\nu(x, X_0) = \tau$ for all $x \in X_0$. By Theorem 2, in X_0 there lies a bicompactum X which is continuously mapped onto D^τ. The theorem is proved.

§ 4. **Depth of images of the generalized Cantor discontinuum.**

Lemma 2. *If a bicompactum X is continuously mapped onto D^τ, then X topologically contains a discrete space T such that $|\overline{T}| \geq 2^\tau$.*

Proof. Let M be some dense subset of D^τ, $|M| = \tau$, T a discrete space, $|T| = \tau$, and $\psi : M \to T$ an arbitrary one-to-one mapping of M onto T. Let $Y = D^\tau \oplus T$ be the disjoint union of D^τ and T. Introduce on Y the following topology. The points of the set T will be considered isolated. For any point $x \in D^\tau$, declare a neighborhood $U(x)$ to be the set

$$U(x) = H(x) \cup$$

$\bigcup \varphi(H(x) \cap M) \setminus \varphi(x)$, if $H(x)$ is a basic neighborhood of x in D^τ. It can be shown that Y in this topology is a zero-dimensional compact extension of T, moreover $|Y| = 2^\tau$ and $wY = \tau$. Since, by N. B. Vedenisov' s theorem, D^τ is universal for all zero-dimensional spaces, we have $Y \subset_{\text{top}} D^\tau$. Let $f : X \to D^\tau$ be a continuous mapping of X onto D^τ. By Brouwer' s theorem $(^8), p.178$, there exists an irreducible preimage Z of the space Y lying in X. Since under an irreducible mapping the preimage of an isolated point is an isolated point, and the preimage of a dense set is dense, Z is a compact extension of T. Since $f(Z) = Y$, it follows that $|Z| \geq 2^\tau$. The lemma is proved.

Theorem 3. *Let R be a dyadic compactum of weight τ, with $cf(\tau) = \aleph_0$. Then $gR = |R|$. In particular, the depth of a metrizable compactum is equal to its cardinality and is attained.*

Theorem 4. *If the weight of a dyadic compactum R satisfies one of the conditions of the main theorem, then the depth of R is equal to its cardinality and is attained.*

Theorem 5. *The depth of any compactum without isolated points is not less than the continuum.*

Theorem 6. *Every dyadic compactum without isolated points contains a Cantor perfect set.*

§ 5. Consider the following three conditions.

(ζ_1). For every ordinal number a we have $2^{\aleph_a} = \aleph_{a+1}$.

(ζ_2). There are no weakly inaccessible cardinals.

(ζ_3). Every weakly inaccessible cardinal is also strongly inaccessible.

The definitions and properties of weakly and strongly inaccessible cardinal numbers we take from A. Tarski (^{9}). The definitions and properties of absolutes of topological spaces we take from V. I. Ponomarev (^{5}). Theorems proved using the conditions (ζ_1), (ζ_2), or (ζ_3) are marked respectively by the letters $\zeta_1, \zeta_2, \zeta_3$.

Lemma 3 (ζ_1 or ζ_3). *If the weight of a dyadic compactum R is a weakly inaccessible cardinal, then there exists in R a nonempty open set $U \subset R$ such that $v(x, R) = wR$ for all $x \in U$ and for any mapping $f : D^\tau \to R$.*

The proof of this lemma is analogous to the proof of Theorem 16 in (^{2}).

Theorem 7 (ζ_1 or ζ_2 or ζ_3). *The depth of any dyadic compactum is equal to its cardinality and is attained.*

The proof of this theorem follows from the main theorem, Theorems 4, 5, Lemmas 3 and 4, and Corollary 4.

Theorem 8 (ζ_1 or ζ_2 or ζ_3). *Every dyadic compactum whose weight is an admissible cardinal, i.e. $(wR)^{\aleph_0} = wR$, topologically contains the absolute D^τ, and consequently (^{3}), all extremally disconnected spaces of weight $\leq wR$.*

Theorem 9 (ζ_1). *The weight of a dyadic compactum* R ($wR \geq \aleph_1$) *is equal to the least upper bound of the weights of Stone-Čech compactifications of discrete spaces lying in it. If* $wR = 2^\tau$, $\tau \geq \aleph_0$, *then the least upper bound is attained.*

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REFERENCES

1. A. V. Arhangel' skii, DAN, **175**, No. 4, 751 (1967).
2. B. Efimov, Trudy Mosk. matem. obshch., **14**, 211 (1965).
3. B. Efimov, DAN, **172**, No. 4, 771 (1967).
4. B. Efimov, DAN, **178**, No. 3, 525 (1968).
5. V. I. Ponomarev, UMN, **21**, No. 4, 101 (1966).
6. R. Engelking, A. Pełczyński, Colloq. Math., **11**, No. 1, 55 (1963).
7. H. A. Shanin, Tr. Matem. inst. im. V. A. Steklova AN SSSR, **24**, 3 (1948).
8. F. Hausdorff, *Set Theory*, Moscow, 1937.
9. A. Tarski, H. Keisler, Fund. Math., **53**, 225 (1964).

Note: Figure translations are in progress. See original paper for figures.