ON A PERIODIC EQUATION OF SECOND ORDER

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Abstract

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MATHEMATICS

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ON A PERIODIC EQUATION OF SECOND ORDER

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The present note continues the investigations published in ([10], [16]).

Let u_0, u_1 $(u_0(0) = 1,\ \dot{u}_0(0) = 0,\ u_1(0) = 0,\ \dot{u}_1(0) = 1)$ be a fundamental system of solutions of the equation

$$\mathcal{L}x \equiv \ddot{x} + q(t)\dot{x} + p(t)x = 0, \qquad t \in (-\infty, \infty), \tag{1}$$

where $q, p \in L[0, \omega]$ are ω-periodic.

By the multipliers of equation (1) (see ([18]), p. 98) we shall mean the roots of the characteristic equation

$$\lambda^2 - [u_0(\omega) + \dot{u}_1(\omega)]\lambda + \exp(-\omega I_q) = 0$$

$$\left(I_q = \frac{1}{\omega} \int_0^\omega q(t)\, dt \right).$$

It is easy to see that a real number λ is not a multiplier if and only if there exists a Green' s function for the problem

$$\mathcal{L}x = 0, \qquad x(\omega) = \lambda x(0), \qquad \dot{x}(\omega) = \lambda \dot{x}(0), \tag{2}$$

i.e., if and only if (2) has the unique trivial solution $x(t) \equiv 0$.

Many questions in the theory of equation (1) reduce to the estimation of multipliers (see ([14])). For example, this equation is asymptotically stable for $t \in [0, \infty)$ if and only if $I_q > 0$ and, for all λ, $|\lambda| \geq 1$, there exists a Green' s function for problem (2).

As usual, we identify two functions that differ only on a set of measure zero, and, in order to shorten formulations, omit qualifications necessary when dealing with summable functions (the inequality holds almost everywhere, etc.).

1. As in ([10], [16]), denote by $\alpha = r_0(\alpha), r_1(\alpha), r_2(\alpha), \ldots$ the successive zeros of a nontrivial solution of equation (1) having its first zero at $t = \alpha$ ($r_i < r_{i+1}$).

Lemma 1. *A Green' s function of problem (2) exists for every $\lambda < 0$ (there are no negative multipliers of equation (1)) if and only if, for all $\alpha \in [0, \omega]$, either $\alpha + \omega < r_1(\alpha)$, or, for some $n = 0, 1, \ldots$,*

$$r_{2n+1}(\alpha) < \alpha + \omega < r_{2n+3}(\alpha).$$

2. Consider problem (2) for $\lambda > 0$. We shall say that **condition (A)** is satisfied if problem (2) has no eigenfunction of constant sign, and **condition (B)** if (2) has no sign-changing eigenfunction. Thus, for the existence of a Green' s function of problem (2), it is necessary and sufficient that conditions (A) and (B) be satisfied simultaneously.

Put

$$\rho = \frac{1}{\omega} \ln \lambda, \qquad p^*(\rho, t) \equiv \rho^2 + \rho I_q + \frac{1}{4} I_q^2 + p(t) - \frac{1}{4} q^2(t) - \frac{1}{2} \dot{q}(t).$$

Lemma 2. *Let $\lambda > 0$. Condition (A) is satisfied if and only if, for some natural k, on the interval $[0, k\omega]$ there exists a funct-*

function $v(t)$ with absolutely continuous first derivative such that
$v(k\omega) = v(0)$, $\dot{v}(k\omega) \leq \dot{v}(0)$ (≥ 0),
$\mathcal{L}_\rho v \equiv \mathcal{L} v + 2\rho \dot{v} + (\rho^2 + \rho q) v \geq 0$
$(\mathcal{L}_\rho v \leq 0)$, and either $\mathcal{L}_\rho v \not\equiv 0$, or $\dot{v}(k\omega) \neq \dot{v}(0)$.

Theorem 1. Suppose that any one of the following conditions is satisfied:

a) $p \not\equiv 0$ on $[0, \omega]$, $I_q \geq -\rho$, and for all $\alpha \in [0, \omega]$

$$I(\alpha) \equiv \int_\alpha^{\alpha+\omega} dt \int_\alpha^t \exp\left(- \int_s^t q(\tau)\, d\tau \right) ds \geq -\rho\omega;$$

b) $p^*(\rho, t) \in L[0, \omega]$, $p^*(\rho, t) \not\equiv 0$ on $[0, \omega]$,

$$\int_0^\omega p^*(\rho, t)\, dt \geq 0.$$

Then for $\lambda = \exp \rho\omega$ condition (A) is satisfied.

Lemma 3. In order that, for every $\lambda > 0$, condition (B) be satisfied, it is necessary and sufficient that, for all $\alpha \in [0, \omega]$ and some $n = 0, 1, \ldots$, the inequalities

$$r_{2n}(\alpha) < \alpha + \omega < r_{2n+2}(\alpha). \tag{3}$$

hold.

Corollary 1. Suppose that for all $\alpha \in [0, \omega]$, $\alpha + \omega < r_2^*(\alpha)$. Suppose, moreover, that one of the following conditions is satisfied:

a) $p \not\equiv 0$ on $[0, \omega]$, $I_q \geq 0$, and for all $\alpha \in [0, \omega]$, $I(\alpha) \geq 0$;

b) $p^*(0, t) \in L[0, \omega]$, $p^*(0, t) \not\equiv 0$ on $[0, \omega]$, $I_q \geq 0$,

$$\int_0^\omega p^*(0, t)\, dt \geq 0.$$

Then the Green' s function of problem (2) exists for all $\lambda \geq 1$.

Remark. If inequalities (3) are satisfied for $n \geq 1$, then for any $\lambda > 0$ the Green' s function of problem (2) exists.

3. To construct effective conditions guaranteeing the inequality $\beta < r_1(\alpha)$, it is convenient to use the well-known Wintner-Levinson nonoscillation criterion ([21]) (see also Corollary 2 of ([1])), namely that $\beta < r_1(\alpha)$ if and only if on $[\alpha, \beta]$ there exists a function $v(t) \geq 0$ such that $\mathcal{L}v \leq 0$, with $\mathcal{L}v \not\equiv 0$ on $[\alpha, \beta]$. From Remark 2 to Theorem 1 of ([3]) we obtain the following oscillation criterion.

Lemma 4. $r_1(\alpha) < \beta$ if and only if on $[\alpha, \beta]$ there exists such a $v(t)$, possessing an absolutely continuous first derivative, that
$\dot{v} \geq 0$, $v(\alpha) = v(\beta) = 0$, $\mathcal{L}v \geq 0$, with $\mathcal{L}v \not\equiv 0$ on $[\alpha, \beta]$.

Theorem 2. Suppose that $q(t) \geq -2m_1 \leq 0$ for $t \in [\alpha, \alpha + T_1]$,
$q(t) \leq 2m_2 \geq 0$ for $t \in [\alpha + T_1, \alpha + T_1 + T_2]$,
$p(t) \geq k > m_i^2$ $(i = 1, 2)$, where

$$T_i = \frac{1}{\sqrt{k - m_i^2}} \left(\pi - \operatorname{arcctg} \frac{m_i}{\sqrt{k - m_i^2}} \right), \qquad i = 1, 2.$$

If $\beta - \alpha > T_1 + T_2$, then $r_1(\alpha) < \beta$.

For the proof of Theorem 2 it suffices to use Lemma 4, taking as the function $v(t)$ the solution of the equation
$\ddot{x} + M(t)\dot{x} + kx = 0$, where $M(t) = -2m_1$ for $t \in [\alpha, \alpha + T_1]$ and $M(t) = 2m_2$ for $t \in [\alpha + T_1, \alpha + T_1 + T_2]$.

For $q \equiv 0$, Theorem 2 gives the well-known oscillation criterion of N. E. Zhukovsky ([5]) (see also ([18])): if $p(t) \geq \pi^2/(\beta - \alpha)^2$ ($\not\equiv$), then $r_1(\alpha) < \beta$.

Analogously, the following assertion is proved, close in content to Corollary 3 of ([9]) (see also ([4,12,13,19])).

Theorem 3. Suppose that $q(t) \leq 2m_1 \geq 0$ for $t \in [\alpha, \alpha + T_1']$,
$q(t) \geq -2m_2 \leq 0$ for $t \in [\alpha + T_1', \alpha + T_1' + T_2']$,
$p(t) \leq l > m_i^2$ $(i = 1, 2)$,

where

$$T_i' = \frac{1}{\sqrt{l-m_i^2}} \operatorname{arcctg} \frac{m_i}{\sqrt{l-m_i^2}}, \qquad i=1,2.$$

If $\beta-\alpha < T_1' + T_2'$, then $r_1(\alpha) > \beta$.

From Theorems 2 and 3 we obtain

Corollary 2. Let $|q(t)| \le 2m,\ m^2 < k \le p(t) \le l$, and for some integers n_1, n_2 $(0 \le n_1 < n_2)$

$$\frac{2n_1}{\sqrt{k-m^2}}\left(\pi - \operatorname{arcctg}\frac{m}{\sqrt{k-m}}\right) \le \beta-\alpha \le \frac{2n_2}{\sqrt{l-m^2}} \operatorname{arcctg}\frac{m}{\sqrt{l-m^2}}.$$

Then

$$r_{n_1}(\alpha) < \beta < r_{n_2}(\alpha). \tag{4}$$

For other effective sufficient conditions guaranteeing inequalities (4), see ([9,11,20]).

4. The linear assertions formulated above prove useful in the study of questions of existence, uniqueness, and stability of periodic solutions of the nonlinear equation

$$\mathcal{L}x \equiv \ddot{x} + q(t)\dot{x} + p(t)x = f(t,x,\dot{x}), \qquad t \in (-\infty,\infty). \tag{5}$$

We shall assume that $f(t,x,y)$ satisfies the Carathéodory smoothness conditions ([6]), is ω-periodic in t, and is nonnegative for $x \ge 0,\ y \in (-\infty,\infty)$.

The following assertion supplements Theorem 1 of ([7]).

Theorem 4. Let the Green' s function of problem (2) for $\lambda = 1$ exist and be positive. Suppose, further, that for $t \in [0,\omega],\ x \ge 0,\ y \in (-\infty,\infty)$, the inequality

$$f(t,x,y) \le a(t)x + b(t)|y|^\gamma + c, \qquad 0 \le \gamma < 1$$

is satisfied.

If the Green' s function of the problem $\mathcal{L}x = a(t)x,\ x(\omega) = x(0),\ \dot{x}(\omega) = \dot{x}(0)$ exists and is positive, then equation (5) has at least one positive ω-periodic solution.

Remark. Theorem 4 remains valid if $f(t,x,y)$ is discontinuous in x, y, satisfies the conditions of ([2]), and the solution is understood in the sense of ([2]).

The following assertion supplements the results of ([10]) on conditions for positivity of the Green' s function of problem (2) for $\lambda = 1$.

Theorem 5. Let the equation $\mathcal{L}x = 0$ be nonoscillatory on $[a, a+\omega]$, i.e., for every $a \in [0,\omega)$, $a + \omega < r_1(a)$. Suppose, further, that $p \neq 0$ on $[0,\omega]$ and at least one of the following conditions is fulfilled:

a) for all $a \in [0,\omega)$, $I(a) \geq 0$;

b) for all $a \in [0,\omega)$,

$$\int_a^{a+\omega} p(t) \exp\left(\int_a^t q(s)\,ds\right) dt \geq 0;$$

c) $p^*(0,t) \in L[0,\omega]$,

$$\int_0^{\omega} p^*(0,t)\,dt \geq 0.$$

Then the Green's function $G(t,s)$ of problem (2) for $\lambda = 1$ exists and $G(t,s) > 0$ in the square $t, s \in [0,\omega]$.

For other conditions guaranteeing positivity of the Green' s function of problem (2) for $\lambda = 1$, see ([8,15,17]).

For the case when the right-hand side of equation (5) does not depend on $\dot{x}$, i.e., for the equation

$$\mathcal{L}x \equiv \ddot{x} + q(t)\dot{x} + p(t)x = f(t,x), \qquad t \in (-\infty,\infty), \tag{6}$$

the following assertion is valid.

Theorem 6 (cf. (7)). Let the Green' s function of problem (2) for $\lambda = 1$ exist and be positive. Suppose, further, that for $t \in [0,\omega]$ and $x_1 \geq x_2 \geq 0$ the inequalities

$$b(t)(x_1 - x_2) \leq f(t,x_1) - f(t,x_2) \leq a(t)(x_1 - x_2)$$

hold. If the Green' s function of the problem $\mathcal{L}x = a(t)x,\ x(\omega) = x(0),\ \dot{x}(\omega) = \dot{x}(0)$ exists and is positive, and for the equation $\mathcal{L}x = b(t)x$ one has $\alpha + \omega < r_2(\alpha)$ for all $\alpha \in [0,\omega)$, then equation (6) has a unique positive ω-periodic solution.

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Note: Figure translations are in progress. See original paper for figures.