CRITERIA FOR THE FUNDAMENTALITY OF PERMUTATION GROUPS AND TRANSFORMATION SEMIGROUPS

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Abstract

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MATHEMATICS

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CRITERIA FOR THE FUNDAMENTALITY OF PERMUTATION GROUPS AND TRANSFORMATION SEMIGROUPS

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Let $k \geq 3$; $E_k = \{0, \dots, k-1\}$; $\mathcal{P}_k$ be the set of all functions of k-valued logic; $P_k = \langle \mathcal{P}_k; \Phi = \{\eta, \tau, \Delta, \nabla, *\}\rangle$ be the Post algebra of rank k (for the definition of the signature operations from Φ, see ([1])); Ω_k be the symmetric semigroup of degree k; σ_k the symmetric group of degree k; A_k the alternating group of degree k (the subgroup of all even permutations in σ_k). Let us denote by $\mathcal{T}$, and call the Słupecki algebra, the subalgebra in P_k generated by all functions depending essentially on only one argument and by all functions omitting at least one value from E_k. Following A. Salomaa ([2]), we shall call a subset $\mathfrak{M} \subset \mathcal{P}_k$ fundamental if the condition is satisfied (prior to the reference to ([3]) we do not assume the maximality of the subalgebra $\mathcal{T}$ in P_k to be known):

$$([\mathfrak{M}] \neq \mathcal{P}_k) \ \&\ (\forall f \in \mathcal{P}_k \setminus (\mathcal{T} \cup [\mathfrak{M}])([\mathfrak{M} + \{f\}] = \mathcal{P}_k)). \tag{1}$$

Since, when testing a set for fundamentality, it must first be closed, we are interested only in the fundamentality of closed sets (subalgebras in P_k). For well-known reasons, special interest attaches to the study of the fundamentality of subalgebras of unary functions (strictly speaking, one studies the fundamentality of subsemigroups of the symmetric semigroup Ω_k); for them condition (1) is transformed into the form

$$\forall f \in \mathcal{P}_k \setminus \mathcal{T}([\mathfrak{M} + \{f\}] = \mathcal{P}_k).$$

It should be noted that the existence of fundamental sets not contained in $\mathcal{T}$ is an obvious fact (such are, for example, all maximal subalgebras in P_k distinct from $\mathcal{T}$); whereas the existence of fundamental semigroups of unary functions is a non-obvious fact. In ([3]) the fundamentality of the semigroup Ω_k was proved (and thereby the existence of fundamental semigroups), and in ([4]) the fundamentality of its ideal $\Omega_{k,k-1}$ (by $\Omega_{k,t}$ we denote the ideal of the semigroup Ω_k consisting of all mappings of rank $\leq t$, $1 \leq t \leq k-1$). After it had been proved in ([5]) that the symmetric group $\sigma_k = \Omega_k \setminus \Omega_{k,k-1}$ is fundamental ($k \geq 5$), the study of fundamental semigroups naturally split into three directions: the study of the fundamentality of permutation groups (subgroups of σ_k), the study of the fundamentality of subsemigroups of the ideal $\Omega_{k,k-1}$ (semigroups having no σ_k-component), and the study of the fundamentality of semigroups having both a σ_k-component and an $\Omega_{k,k-1}$-component (we shall call such semigroups mixed). From ([6]) it follows that the ideal $\Omega_{k,k-2}$ (and consequently any of its subsemigroups) is no longer a fundamental semigroup, and among the semigroups X satisfying the condition $\Omega_{k,k-2} \subset X \subset \Omega_{k,k-1}$ there are both fundamental and nonfundamental ones. From ([4],[5] ,[7]) it follows that for $k \geq 5$ all maximal subgroups in Ω_k are fundamental, and that the unique nonfundamental semigroup among the maximal subsemigroups in Ω_q ($q = 3, 4$) is $\Omega_{q,q-2} + \sigma_q$. In ([8]) the fundamentality was proved

groups A_k ($k \geqslant 5$), in [2]—for $k \geqslant 5$ the fundamentality of every 4-fold transitive permutation group, and for $5 \leqslant k \neq 2^i$ the fundamentality of every 3-fold transitive one. A. I. Mal' cev in [9] proved the fundamentality of every $(k-1)$-fold transitive subsemigroup in Ω_k ($k \geqslant 5$), which absorbs the results of [3, 4, 5], but does not absorb the results of [8, 2]. The author has not encountered other investigations on fundamental sets in the literature.

In the present note, for "almost all" k, a necessary and sufficient criterion is given for the fundamentality of permutation groups (as a consequence of the latter, a sufficient criterion is derived for the fundamentality of mixed semigroups of mappings), and the possibility of extending to all k the method of its proof is also analyzed. Before turning to the formulations of the assertions, we introduce the necessary notation.

By N we denote the natural series, by N_i the set of i-th powers of all numbers, by P the set of all prime numbers. Let

$$\widetilde{N} = N \setminus (P \cup N_2 \cup N_3 \cup \ldots).$$

For every algebra $\mathfrak{A}$, by $\mathrm{Sub}(\mathfrak{A})$ we denote the set of all its subalgebras, and by $\mathrm{Sub}^{\max}(\mathfrak{A})$ the set of all its maximal subalgebras (the latter, by definition, are regarded as proper). If $\mathfrak{R} \subseteq \mathrm{Sub}(\mathfrak{A})$, then by $\mathfrak{R}^{\max}$ we denote the collection of all maximal elements of the set $\mathfrak{R}$, partially ordered by set-theoretic inclusion, and by $\mathfrak{R}^{\mathrm{Sub}}$ the collection of all subalgebras of all algebras from $\mathfrak{R}$. The fundament of a subalgebra $\mathfrak{A} \in \mathrm{Sub}(P_k)$ is called its σ_k-component; denote it by $\mathrm{Fund}(\mathfrak{A})$. If $\mathfrak{M} \subseteq \mathrm{Sub}(P_k)$, then

$$\mathfrak{M}^{\mathrm{Fund}} = \{X : X \in \mathrm{Sub}(\sigma_k),\ \exists Y \in \mathfrak{M}\ (X = \mathrm{Fund}(Y))\}.$$

If τ is some operator mapping $\mathfrak{R} \subseteq \mathrm{Sub}(\sigma_k)$ onto some subset of it, then by $\mathfrak{R}^{\tau}$ we denote the range of the operator τ. In particular, the set of all imprimitive subgroups in σ_k will be denoted by $\mathrm{Sub}^{\mathrm{impr}}(\sigma_k)$, the set of all linear subgroups in σ_k by $\mathrm{Sub}^{\mathrm{lin}}(\sigma_k)$ (a subgroup $X \in \mathrm{Sub}(\sigma_k)$ is called linear if either $X = L_k$, or X is conjugate to L_k, where L_k is the subgroup of all permutations from σ_k of the form $\psi(x) = \alpha + \beta x \pmod{k}$). Finally, by $\mathrm{Sub}^{\mathrm{bas}}(\sigma_k)$ we denote the set of all fundamental subgroups in σ_k. Everywhere below $k \geqslant 5$, since for $k \leqslant 4$ no fundamental groups exist.

Theorem 1. If $k \in \widetilde{N}$,

$$(\mathrm{Sub}^{\max}(P_k) \setminus \{\mathcal{T}\})^{\mathrm{Fund}} \subseteq \mathrm{Sub}^{\mathrm{impr}}(\sigma_k),$$

while if $k \in P$,

$$(\mathrm{Sub}^{\max}(P_k) \setminus \{\mathcal{T}\})^{\mathrm{Fund}} \subseteq \mathrm{Sub}^{\mathrm{impr}}(\sigma_k) \cup (\mathrm{Sub}^{\mathrm{lin}}(\sigma_k))^{\mathrm{Sub}}.$$

Theorem 2.

$$(\mathrm{Sub}^{\mathrm{impr}}(\sigma_k))^{\max} = (\mathrm{Sub}^{\max}(\sigma_k))^{\mathrm{impr}} \quad \text{for all } k,$$

while if $k \in P$,

$$((\mathrm{Sub}^{\mathrm{lin}}(\sigma_k))^{\mathrm{Sub}})^{\max} = (\mathrm{Sub}^{\max}(\sigma_k))^{\mathrm{lin}\cdot\mathrm{Sub}} = (\mathrm{Sub}^{\max}(\sigma_k))^{\mathrm{lin}} = \mathrm{Sub}^{\mathrm{lin}}(\sigma_k).$$

Corollary. If $k \in \widetilde{N}$,

$$(\mathrm{Sub}(\sigma_k) \setminus \mathrm{Sub}^{\mathrm{bas}}(\sigma_k))^{\max} = \mathrm{Sub}^{\max}(\sigma_k) \setminus \mathrm{Sub}^{\mathrm{bas}}(\sigma_k).$$

Theorem 3. If $k \in \widetilde{N}$, the fundamental subgroups in σ_k are exhausted by its primitive subgroups:

$$\mathrm{Sub}^{\mathrm{bas}}(\sigma_k) = \mathrm{Sub}(\sigma_k) \setminus \mathrm{Sub}^{\mathrm{impr}}(\sigma_k);$$

if $k \in P$, they are the primitive subgroups that are not contained in any of the linear ones:

$$\mathrm{Sub}^{\mathrm{bas}}(\sigma_k) = \mathrm{Sub}(\sigma_k) \setminus (\mathrm{Sub}^{\mathrm{impr}}(\sigma_k) \cup (\mathrm{Sub}^{\mathrm{lin}}(\sigma_k))^{\mathrm{Sub}}).$$

Corollary. If $k \in \widetilde{N}$, every mixed subsemigroup in Ω_k with primitive σ_k-component is fundamental. If $k \in P$, fundamental will be every mixed subsemigroup in Ω_k whose σ_k-component is a primitive subgroup not contained in any of the linear ones.

Theorem 4.

$$(\mathrm{Sub}(\sigma_k) \setminus \mathrm{Sub}^{\mathrm{bas}}(\sigma_k))^{\max} = \mathrm{Sub}^{\max}(\sigma_k) \setminus \mathrm{Sub}^{\mathrm{bas}}(\sigma_k) \Longleftrightarrow$$

$$\Longleftrightarrow (\mathrm{Sub}^{\max}(P_k) \setminus \{\mathcal{T}\})^{\mathrm{Fund}} = \bigcup_{\tau \in \mathfrak{M} \subseteq Z_{\mathfrak{A}}(\max) + \mathfrak{B}} \mathrm{Sub}^{\tau}(\sigma_k),$$

where

$$\mathfrak{A} = \{\tau : \mathrm{Sub}(\sigma_k) \xrightarrow{\tau} \mathrm{Sub}(\sigma_k) \setminus \mathrm{Sub}^{\mathrm{bas}}(\sigma_k)\},$$

$$Z_{\mathfrak{A}}(\max) = \{\tau : \tau \in \mathfrak{A},\ (\mathrm{Sub}^{\tau}(\sigma_k))^{\max} = (\mathrm{Sub}^{\max}(\sigma_k))^{\tau}\},$$

$$\mathfrak{B} = \{\tau : \tau \in \mathfrak{A} \setminus Z_{\mathfrak{A}}(\max),\ \mathrm{Sub}^{\tau}(\sigma_k) \subseteq \mathrm{Sub}^{\max}(\sigma_k) \setminus \mathrm{Sub}^{\mathrm{bas}}(\sigma_k)\}.$$

It is obvious that, for those

$$k \in N \setminus (\widetilde{N} \cup P) = \bigcup_{i \geq 2} N_i,$$

for which $(\mathrm{Sub}_{\max}(P_k) \setminus \{\mathcal{T}\})^{\mathrm{Fund}}$ is representable in the form of the indicated union, Theorem 4 gives a method for obtaining a necessary and sufficient criterion for the fundamentality of subgroups in σ_k. Let us note that the set $\widetilde{N} \cup P$ for which this has been done (Theorem 3) constitutes "almost all" of the natural numbers.

In conclusion we note that it is easy to formulate theorems dual to Theorems 3 and 4 for the corresponding predicate algebras (the set of all predicates defined on E_k can be turned into an algebra so that the lattice of its subalgebras is anti-isomorphic to the lattice of subgroups of σ_k (see ([10]))).

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Note: Figure translations are in progress. See original paper for figures.