EQUIVALENCE GROUPS OF LINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER

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Abstract

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EQUIVALENCE GROUPS OF LINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER

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1. Let Φ be an n-dimensional arithmetic space or some domain in it; x^α are the coordinates of a point $x \in \Phi$ $(\alpha = 1, \dots, n)$, and $A^{\alpha\beta} = A^{\beta\alpha}$, B^α, C, V, for fixed α, β, are square matrices of order m, defined and sufficiently smooth in Φ. Greek (Latin) indices are assumed to range from 1 to n (m), $\psi = (\psi^1, \dots, \psi^n)$, f, F, D are column vectors, which may be regarded as square matrices defined in Φ and having m identical columns. Let $\partial_\alpha = \partial/\partial x^\alpha$, $\partial_{\alpha\beta} = \partial_\alpha\partial_\beta$, $u^\beta_\alpha = \partial_\alpha \psi^\beta$ $(\alpha, \beta = 1, \dots, n)$, $L = A^{\alpha\beta}\partial_{\alpha\beta} + B^\alpha\partial_\alpha + C$. By $\|A^{r\alpha\beta}_k\|^{mn}_1$ we denote the matrix of order mn, obtained by writing, instead of each element of the matrix $A^{\alpha\beta}$ $(\alpha, \beta$ fixed), a matrix of order n, making α, β range from 1 to n. Suppose that the determinant Δ of the matrix $\|A^{r\alpha\beta}_k\|^{mn}_1$ is nonzero inside Φ, and construct the inverse matrix $\|A^r_{k\alpha\beta}\|^{mn}_1$, i.e.

$$A^{r\alpha\beta}_k A^k_{i\beta\gamma} = \delta^r_i \delta^\alpha_\gamma \quad (i, r = 1, \dots, m;\ \alpha, \gamma = 1, \dots, n).$$

Consider the vector differential equation

$$Lf = D \tag{1}$$

with certain additional conditions $K(f)$, ensuring the uniqueness of its solution f for any given matrix D. In addition, there is a domain Φ', formed by the points $x' = (x'^1, \dots, x'^n)$, matrices $A'^{\alpha\beta}$, B'^α, C', D', conditions $K'(f')$, an operator L', and the equation

$$L'f' = D', \tag{1'}$$

in which f' is the sought column vector.

Definition 1. Equations (1) and (1′) are called **equivalent** if there exist matrices ψ, V, F, W, defined in Φ, with continuous second derivatives, such that the equalities

$$x' = \psi(x), \quad f = Vf' + F, \tag{2}$$

$$\Phi' = \psi(\Phi), \quad K'(f') = K(Vf' + F), \quad A'^{\alpha\beta} = WA^{\gamma\delta}u^{\alpha}_{\gamma}u^{\beta}_{\delta}V,$$

$$B'^{\alpha} = W[u^{\alpha}_{\beta}B^{\beta}V + A^{\beta\gamma}(u^{\alpha}_{\beta\gamma}V + 2u^{\alpha}_{\beta}\partial_{\gamma}V)], \tag{3}$$

$$C' = WLV, \quad D' = W(D - LF),$$

hold, and the Jacobian of the change of variables is nonzero and these formulas are uniquely solvable with respect to x, f', and the unprimed coefficients $(u^{\gamma}_{\alpha\beta} = \partial_{\alpha}u^{\gamma}_{\beta})$. If, instead of Φ and Φ', neighborhoods of the points x and x', respectively, filling Φ and Φ', occur in this definition, and the second equality in (3) is absent, then we obtain locally equivalent equations (1) and (1′).

The aim of the article is to investigate the equivalence conditions in the case when the matrices L and L' are symmetric. If ψ, V, F, W are regarded as given, then the transformation (2)+(3) takes equation (1) to an equivalent equation (1′). The set G of all such transformations forms a group, wh-

we shall call such a group the equivalence group of equation (1). In essence, (2) + (3) are equations of the group G; however, even with fixed primed coefficients, finding the elements of this group in general form is difficult. In this connection, special groups generated by equations of the form (1) are studied (for example, see (^{1})). All group properties of such equations are in one way or another derived from tensor-invariant equivalence conditions. Such conditions were obtained in (^{2}) for the case $m = V = 1$, $C = C' = F = 0$. However, for a number of reasons evident from the text of the article, the method of (^{2}) cannot be applied to the general case.

2. For $n > 2$, denote by $A_{\alpha\beta}$ the matrix of order m with elements $A^{r}_{k\alpha\beta}$ (α, β fixed, $k, r = 1, \ldots, m$). Let $\Gamma^{\gamma}_{\alpha\beta}$, $C^{\delta}_{\alpha\beta\gamma}$, $R^{\delta}_{\alpha\beta\gamma}$, $R_{\alpha\beta}$, R be, respectively, the Christoffel symbols of the second kind, the tensors of conformal curvature, curvature, Ricci, and scalar curvature, obtained from $A_{\alpha\beta}$ as from a metric tensor

$$u'^{\beta}_{\alpha} = \partial'_{\alpha}x'^{\beta}, \qquad D^{\alpha} = B^{\alpha} + \Gamma^{\alpha}_{\beta\gamma}A^{\beta\gamma}, \qquad D_{\alpha} = D^{\beta}A_{\alpha\beta}, \qquad \varphi = WV,$$

$$r_{\alpha\beta} = \frac{1}{n-2}\left(2R_{\alpha\beta} - \frac{1}{n-1} R A_{\alpha\beta}\right),$$

$$g = 4C - D_\alpha D^\alpha - 2(\nabla_\alpha D_\beta) A^{\alpha\beta} - (n-2)(n-1)^{-1} R,$$

$$\overline{\Gamma}^\beta_{\varepsilon\omega} = \Gamma^\beta_{\varepsilon\omega} + T^\beta_{\varepsilon\omega}, \qquad T^\beta_{\varepsilon\omega} = \frac{1}{2}\left(A^{\beta\alpha} A_{\varepsilon\omega} \partial_\alpha\varphi - \delta^\beta_\omega \partial_\varepsilon \varphi - \delta^\beta_\varepsilon \partial_\omega \varphi\right)\varphi^{-1};$$

we shall give analogous meaning to the primed quantities. By ∇_α, $\overline{\nabla}_\alpha$ are denoted the symbols of covariant differentiation with respect to $\Gamma^\gamma_{\alpha\beta}$ and $\overline{\Gamma}^\gamma_{\alpha\beta}$, respectively. Form the equation

$$\nabla_\alpha(\varphi^{-1}\nabla_\beta\varphi) + \frac{1}{2}\left(\nabla_\alpha\varphi \cdot \nabla_\beta\varphi - \frac{1}{2} A_{\alpha\beta} A^{\omega\gamma} \nabla_\omega\varphi \cdot \nabla_\gamma\varphi\right)\varphi^{-2} = r_{\alpha\beta} - r'_{\gamma\delta} u^\gamma_\alpha u^\delta_\beta, \tag{4}$$

as well as the algebraic system of equations

$$A'^{\alpha\beta} = \varphi A^{\gamma\delta} u^\alpha_\gamma u^\beta_\delta, \qquad u^\beta_\alpha C^\alpha_{\omega\eta\varepsilon} = C'^\beta_{\gamma\sigma\alpha} u^\gamma_\omega u^\sigma_\eta u^\alpha_\varepsilon,$$

$$\varphi g = g', \qquad \overline{\nabla}_{[\alpha} D_{\gamma]} = u^\tau_\alpha u^\varepsilon_\gamma \overline{\nabla}'_{[\tau} D'_{\varepsilon]}, \tag{5}$$

$$\overline{\nabla}_{[\alpha} r_{\beta]\gamma} = u^\delta_\beta u^\varepsilon_\alpha \overline{\nabla}'_{[\varepsilon} r'_{\delta]\omega} u^\omega_\gamma + r'_{\delta\omega} u^\delta_\beta u^\varepsilon_\alpha T^\omega_{\varepsilon\gamma}[\alpha, \beta],$$

whose unknowns are x'^α, u^β_α, φ^r_k, $\partial_\alpha \varphi^r_k$ $(\alpha, \beta = 1, \dots, n;\ k, r = 1, \dots, m)$. We shall differentiate each of the relations (5) an unlimited number of times; here the second derivatives of φ are eliminated by means of (4), and the second derivatives of x'^α by means of the equality

$$u^\rho_{\varepsilon\omega} = u^\rho_\alpha \overline{\Gamma}^\alpha_{\varepsilon\omega} - \Gamma'^\rho_{\alpha\gamma} u^\alpha_\varepsilon u^\gamma_\omega.$$

Denote by $(\mathbb{R})$ the algebraic system of equations that includes (5) and the equalities obtained from (5) in the described way.

Theorem 1. *If $\Delta'\Delta \neq 0$, equations (1) and (1′) are locally equivalent with symmetric matrices V, W if and only if there exists a solution of the system $(\mathbb{R})$ for which the matrix φ is symmetric,*

$$|u^\beta_\alpha|^n_1 \, |\varphi^r_k|^m_1 \neq 0,$$

and the matrix

$$A_\alpha = \frac{1}{2}\left[D'_\varepsilon u^\varepsilon_\alpha - D_\alpha + (1-n/2)\varphi^{-1}\partial_\alpha\varphi\right] \qquad (\alpha = 1, \dots, n)$$

is moreover potential. When these conditions are fulfilled, the matrices V, W *are written explicitly in the form* $V = V_0 \exp\left\{\int_K A_\alpha dx^\alpha\right\}$, $W = \varphi V^{-1}$, *and* F *is a solution of the equation* $LF = D - W^{-1}D'$. *Here* V_0 *is a constant symmetric matrix, and* K *is a piecewise smooth curve in* Φ *connecting the fixed point* x_0 *with the variable point* x $(x_0, x \in \Phi)$.

3. If $V = E$ or $\varphi = E$, then the formulation of Theorem 1 can be modified so that a certain departure from tensorial form, introduced by ···

the following equation in (5), is excluded ($E = \|\delta^k_i\|^m_1$). If $n = 2$, then the equivalence conditions are to some extent simpler; however, they cannot be derived from the case considered.

Let now $n = 1$. Introduce the notation:

$$x^1 = x, \quad A^{11} = A, \quad B^1 = B, \quad A_{11} = A^{-1}, \quad \Gamma = \Gamma^1_{11}, \quad D^1 = D, \quad \partial_1 = \partial/\partial x,$$

$$L = A\partial_{11} + B\partial_1 + C, \quad g = A^{-1}DD + 2A\nabla_1(A^{-1}D) - 4C.$$

We shall differentiate the equations indefinitely

$$\frac{A'}{\sqrt[m]{\Delta'}} = \frac{\varphi A}{\sqrt[m]{|\varphi|\Delta}}, \qquad \nabla'_1 \sqrt[2m]{\Delta'} = \sqrt[2m]{|\varphi|}\left\{\frac{\nabla_1 \sqrt[2m]{\Delta}}{\sqrt[2m]{\Delta}} + \frac{|\varphi|^{-1}\partial_1|\varphi| - m\varphi^{-1}\partial_1\varphi}{2m}\right\}, \tag{6}$$

eliminating the derivatives of u^1 and $\partial_{11}\varphi$ by means of the equalities

$$u^1_1 = \sqrt[2m]{\Delta'}/|\varphi|\Delta, \qquad \partial_{11}\varphi = A^{-1}(g\varphi - g') + \Gamma\partial_1\varphi + \frac{3}{4}\varphi^{-1}(\partial_1\varphi)^2.$$

Assuming $\Delta > 0,\ \Delta' > 0$, denote the union of all equations by (R).

Theorem 2. *The equations* $Lf = 0$ *and* $L'f' = 0$ *are equivalent if and only if the algebraic system* (R) *has a solution* $x', \varphi, \partial_1\varphi,\ |\varphi| > 0$, *for which* $\Phi' = x'(\Phi),\ K'(f') = K(Vf)$.

If we introduce the notation

$$p = \sqrt[2m]{\Delta^{-1}}\partial_1 \sqrt[2m]{\Delta}, \qquad q = A^{-1}g - 2\nabla_1 \frac{\nabla_1 \sqrt[2m]{\Delta}}{\sqrt[2m]{\Delta}} - \left(\sqrt[2m]{\Delta^{-1}}\nabla_1 \sqrt[2m]{\Delta}\right)^2,$$

then the following will hold.

Corollary. *The equation* $Lf = 0$, *by the formulas* $(2) + (3)$, *is reduced to the form* $H'\partial_{11} f' = 0$, *where* H' *is a constant nonsingular matrix, if and only if* q *is a diagonal matrix and the equation of Riccati type* $\partial_1 y + \frac{1}{4} y^2 + py = q$ *has a solution defined in* Φ.

Theorems 1, 2, and also the corollary can be applied to the problem of explicitly constructing solutions of elliptic and hyperbolic equations and systems of second order.

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Note: Figure translations are in progress. See original paper for figures.

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