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1969

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Abstract

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UDC 517.53

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On Two Metric Methods for Defining a Prime End of a Sequence of Plane Domains

(Presented by Academician M. A. Lavrent' ev on 3 January 1969)

In the work (^{1}), M. A. Lavrent' ev introduced the concept of relative distance in a plane domain, by means of which the nature of the continuity of a conformal mapping is clearly revealed. This concept makes it possible to introduce metrically the notion of a prime end in the sense of Carathéodory. At the same time S. Mazurkiewicz (^{2}) metrized the space of prime ends of a simply connected domain B (without, of course, invoking Riemann' s theorem on the existence of a conformal mapping of B onto a disk). G. D. Suvorov (^{3}) defined otherwise and systematically used the metric of prime ends for studying the family $\widehat{BL}$ of more general mappings.

In the article (^{4}) G. D. Suvorov defined the prime ends of a sequence of plane simply connected domains (B_i) converging to a kernel B_0. From the point of view of boundary correspondence under conformal, quasiconformal, and certain more general mappings (see (^{3,5})), the set of these prime ends is the natural boundary of the sequence of domains.

Here we introduce in two ways the notion of a prime end of a sequence of domains (not necessarily simply connected), using conformally invariant metrics in the domains B_i. In this case the fact of boundary correspondence by such prime ends under a univalent conformal mapping $f_i : B_i \to D_i$ of the sequence (B_i) becomes obvious. (On this matter there is our brief communication (^{6}).)

1. Let the domains G and H, containing the origin, be bounded in the planes R_z and R_w, respectively. Below the following notation is used: δ is the Euclidean metric; $\operatorname{Fr} M$ is the boundary of the set M; $\overline{M}$ is the closure of M in the plane; U_i^t is the component containing the point $z = 0$ of the set $\{z \in B_i : \delta(z, R_z \setminus B_i) > t\}$.

We define the distance between nonempty closed subsets $M, N \subset \overline{G}$ as follows:

$$\beta(M, N) = \iint_G |\delta(z, M) - \delta(z, N)| \, dx \, dy.$$

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It can be proved that in the case under consideration β is equivalent to the Hausdorff distance ([7], p. 166) between the sets M and N.

Let $\mathfrak{B}$ be the class of all domains $B_i \subset G$ covering the disk $|z| < \varepsilon$. In the notes ([8,9]) it was proved that the space $\mathfrak{B}$ (with kernel convergence) is metrizable by means of the distance

$$r(B_1, B_2) = \int_0^{\varepsilon} \beta(U_1^t, U_2^t)\, dt.$$

As is known (see ([10,11])), in a domain $B_i \subset G$ one can establish a metric $\rho_i(z, \zeta)$, invariant under conformal mappings of B_i and equivalent to the metric $\delta(z, \zeta) = |z - \zeta|$, such that, if B_i is finitely connected, then its completion $\widetilde{B}_i$ with respect to ρ_i is identifiable with the space of Carathéodory prime ends of the domain B_i.

In what follows we shall assume that $\rho_i(z, \zeta)$ are uniformly bounded with respect to $z, \zeta \in B_i$ and $B_i \in \mathfrak{B}$; otherwise ρ_i should be replaced by the metric $\min(1, \rho_i)$.

2. Denote by b_i an arbitrary element of the completion $\dot{B}_i$ of the domain B_i with respect to ρ_i, and by d_i an arbitrary element of D_i. Let

$$\Sigma = \bigcup_{B_i \in \mathfrak{B}} \dot{B}_i.$$

(Obviously, from $B_i = B_j$ it does not follow that $b_i = b_j$, but the converse is true.)

Define convergence in the set Σ as follows.

Definition*. A sequence $(b_n)_{n=1,2,\ldots} \subset \Sigma$ **converges** to $b_0 \in \Sigma$ if: 1) $B_n \to B_0$ (in $\mathfrak{B}$); 2) for every $\eta > 0$ there exists a point $z_0 \in B_0$ at distance $\rho_0(z_0, b_0) < \eta$ such that $\rho_n(z_0, b_n) < \eta$ for $n > N(z_0)$.

Let, in what follows, f_i be a schlicht conformal mapping of B_i onto $D_i \subset H$, continued to a homeomorphism of $\dot{B}_i$ onto $\dot{D}_i$, with $f_i(0) = 0$, $f_i'(0) > 0$.

Proposition 1. Let $B_n \to B_0$ and let $(f_n)_{n=1,2,\ldots}$ converge uniformly inside B_0 to the function f_0 (necessarily mapping B_0 schlichtly onto D_0, see ([12]), p. 230). If, moreover, $d_n = f_n(b_n)$ and $d_0 = f_0(b_0)$, then the relations $b_n \to b_0$ and $d_n \to d_0$ are equivalent.

Since $D_n \to D_0$ (([12]), p. 230), this proposition is an obvious consequence of the invariance of the metric ρ.

By a Jordan domain we shall mean a domain bounded by a finite number of Jordan curves having no common points.

If one regards a Jordan domain B_0 as a sequence of coincident domains and takes into account the equivalence of ρ_0 and δ for such a domain in $\dot{B}_0$, then from the definition of convergence of b_n to b_0 it easily follows that

Proposition 2. If $B_0 \in \mathfrak{B}$ is a Jordan domain, then the relations $b_0^j \to b_0$ and $|b_0^j - b_0| \to 0$ are equivalent (here b_0, b_0^j are identified with points of the plane).

Proposition 3 (follows from 1 and 2). Let $(f_n)_{n=1,2,\dots}$ converge uniformly to f_0 inside the bounded Jordan domain B_0 (here $f_n : B_0 \to D_n$). If $d_{n_j} = f_{n_j}(b_0^j)$ and $d_0 = f_0(b_0)$, then the relations $|b_0^j - b_0| \to 0$ and $d_{n_j} \to d_0$ are equivalent.

The theorem of G. D. Suvorova (4) on the correspondence of boundaries under a conformal mapping of a sequence of simply connected domains allows one to assert that, when $B_n \to B_0$, the space

$$\sigma = \bigcup_{n=0}^{\infty} \dot{B}_n$$

may be identified with the space of prime ends of the sequence of domains (B_n).

3. The following theorem shows that the concept of a prime end of a sequence (B_n) can also be introduced by means of a suitable metrization of the space $\sigma \subset \Sigma$.

Let

$$\varphi(z,t;b_i) = \begin{cases} \rho_i(z,b_i), & \text{for } z \in U_i^t, \\ 0, & \text{for } z \notin U_i^t. \end{cases}$$

Theorem. Let $\mathfrak{B}_0$ be some equivalence class in $\mathfrak{B}$ with respect to normalized schlicht conformal mappings, i.e., for any $B \in \mathfrak{B}_0$ there exists a conformal mapping f onto one and the same domain D, with $f(0) = 0$, $f'(0) > 0$. Then the space

$$\Sigma_0 = \bigcup_{B_i \in \mathfrak{B}_0} \dot{B}_i$$

is metrizable by means of the distance

$$\widehat{b_1 b_2} = \int_0^{\varepsilon} \iint_G |\varphi(z,t;b_1) - \varphi(z,t;b_2)| \, dx\, dy\, dt + r(B_1, B_2).$$

4. In conclusion we shall show how the conformally invariant metric ρ mentioned in 1 can be defined on the basis of a more general (see (13)) definition of extremal length than in (11).

* Compare (^{3}), p. 219, Lemma 9.

We shall call a curve a continuous mapping γ_n of an interval I (open, half-open, or closed) into a domain B. Let $\Gamma_n = \gamma_n(I)$. A finite collection $\gamma = (\gamma_n)_{1\le n\le N}$ of curves in B is called a generalized curve in B.

We shall call a generalized curve γ a net if the set

$$\Gamma = \bigcup_{n=1}^{N} \Gamma_n$$

is connected and closed relative to B.

By B_0^Γ we denote the component of connectedness of the set $B \setminus \Gamma$ relative to the point $z = 0$. Let B^Γ be the union (possibly empty) of all components C of the set $B \setminus \Gamma$ having the property: $(\operatorname{Fr} C) \setminus \overline{\Gamma}$ includes some component of $\operatorname{Fr} B$.

We shall say that a net γ separates the points z_1 and z_2 of B, if: 1) $z_1 z_2 \subset B_0^\Gamma$ and 2) $z_1, z_2 \in B^\Gamma$ when $\overline{\Gamma} \cap \operatorname{Fr} B = \emptyset$. The family of all possible nets γ separating the points z_1 and z_2 will be denoted by $\gamma_{z_1 z_2}$. Let $\Gamma_{z_1 z_2} = (\Gamma)_{\gamma\in\gamma_{z_1 z_2}}$.

Lemma. If $\Gamma' \in \Gamma_{z_1 z_2}$, $\Gamma'' \in \Gamma_{z_2 z_3}$, but Γ', $\Gamma'' \in \Gamma_{z_1 z_3}$, then $\Gamma' \cap \Gamma'' \ne \emptyset$ and $\Gamma' \cup \Gamma'' \in \Gamma_{z_1 z_2}$.

We define the metric in B by the equality:

$$\rho(z_1, z_2) = \lambda^{1/2}(\gamma_{z_1 z_2}),$$

where λ is the extremal length of the family $\gamma_{z_1 z_2}$ (^{13}, ^{14}).

It is easy to verify that in a neighborhood of each point $z_0 \in B \setminus \{0\}$ the inequality

$$\frac{|z - z_0|}{\sqrt{\pi}} \le \rho(z, z_0) \le \sqrt{2\pi}\,\ln^{-1/2}\frac{a(z_0)}{|z - z_0|}, \qquad a(z_0) > 0, \tag{1}$$

holds, which implies the equivalence of ρ and δ in $B \setminus \{0\}$.

If D is a domain whose boundary consists of a finite number of points or circles without common points, then it is not difficult to see that, for each point $w_0 \in \operatorname{Fr} D$,

$$|w_1 - w_2|/\sqrt{\pi} \le \rho(w_1, w_2) \le \sqrt{2\pi}\,\ln^{-1/2} a(w_0)/\max_{j=1,2}|w_j - w_0| \tag{2}$$

for sufficiently small $a(w_0)$ and $w_1, w_2 \in \{w \in D:\ |w - w_0| < b < a(w_0)\}$.

Since ρ is a conformal invariant, it follows from (1) and (2) that $\rho(z_1, z_2)$ is consistent with the Carathéodory topology in $B\setminus\{0\}$ for every finitely connected

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domain $B \in \mathfrak{B}$. We note that excluding zero does not violate the validity of the propositions formulated in items 2 and 3.

Replacing the Euclidean metric δ by the spherical metric, it is easy to dispense with the condition of uniform boundedness of the domains $B \in \mathfrak{B}$ with respect to δ (see item 1).

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Received
23 XII 1968

CITED LITERATURE

[1] M. A. Lavrent' ev, DAN, 4, No. 5, 207 (1936).
[2] S. Mazurkiewicz, Fund. Math., 6, 272 (1936).
[3] G. D. Suvorov, *Families of Plane Topological Mappings*, Novosibirsk, 1965.
[4] G. D. Suvorov, Matem. sborn., 33 (75), 1, 73 (1953).
[5] B. P. Kufarev, DAN, 181, No. 2, 282 (1968).
[6] P. P. Kufarev, B. P. Kufarev, Abstracts of Brief Scientific Reports, International Mathematical Congress, Section 4, Moscow, 1966, p. 61.
[7] F. Hausdorff, *Set Theory*, Moscow-Leningrad, 1937.
[8] B. P. Kufarev, Tr. Tomskogo gos. univ., 169, mathematical series, issue 1, 3 (1963).
[9] B. P. Kufarev, Tr. Tomskogo gos. univ., 189, mathematical series, issue 3, 133 (1965).
[10] A. D. Myshkis, Matem. sborn., 25 (67), 3, 387 (1949).
[11] I. S. Gal, Proc. Nat. Acad. Sci. U.S.A., 45, 1629 (1959).
[12] T. M. Goluzin, *Geometric Theory of Functions of a Complex Variable*, Moscow, 1966.
[13] E. G. Schlesinger, Am. J. Math., 80, 83 (1958).
[14] L. Ahlfors, A. Beurling, Acta Math., 83, 101 (1950).

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