# ON RIEMANNIAN SPACES WITH THE EUCLIDEAN ISOPERIMETRIC INEQUALITY

V. K. IONIN

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## Abstract

## Full Text

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### MATHEMATICS

**V. K. IONIN**

# ON RIEMANNIAN SPACES WITH THE EUCLIDEAN ISOPERIMETRIC INEQUALITY

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In the present note we consider only twice continuously differentiable Riemannian spaces homeomorphic to Euclidean spaces.

**1°.** By a body in the $n$-dimensional Euclidean space $E^n$ we shall mean the closure of a bounded connected open domain. In an $n$-dimensional Riemannian space $V^n$ we shall call a body the image of any body from $E^n$ under a homeomorphic mapping of $E^n$ onto $V^n$. We shall say that the Euclidean isoperimetric inequality holds in $V^n$ if for every body in $V^n$ there is a body in $E^n$ with no smaller volume and with no larger area (i.e., $(n-1)$-dimensional volume) of the boundary. We denote by $I_n$ the set of all such $n$-dimensional ($n \geq 2$) Riemannian spaces. It is known (see ([1,2])) that $I_2$ consists only of two-dimensional Riemannian spaces with nonpositive Gaussian curvature. For $n \geq 3$ we have proved the following.

**Theorem.** *If in an $n$-dimensional Riemannian space one can introduce a semigeodesic coordinate system $(x^1, \dots, x^n)$ with fundamental form*

$$ds^2 = g_{ij} dx^{idx^j} + (dx^n)^2 \qquad (i, j = 1, \dots, n-1)$$

*so that each hypersurface $x^n = \text{const}$ belongs to $I_{n-1}$, and the determinant*

$$|g_{ij}| = \lambda(x^n)\mu(x^1, \dots, x^{n-1}), \tag{*}$$

*where $\lambda > 0$, $\mu > 0$, and*

$$\left(\frac{1}{\lambda^{1/(2n-2)}}\right)'' \geq 0, \tag{**}$$

*then $V^n \in I_n$, i.e., in $V^n$ the isoperimetric inequality of the Euclidean space $E^n$ holds.*

Condition $(*)$ means that geodesics orthogonal to the hypersurfaces $x^n = \text{const}$ establish between any such hypersurfaces a homeomorphic mapping under which the ratio of the volume of any $(n-1)$-dimensional body to the volume of its image is constant.

The geometric meaning of condition $(**)$ is somewhat clarified by means of the following, easily verified assertion. Let some $(n-1)$-dimensional Riemannian space $V^{n-1}$ with fundamental form

$$ds^2 = a_{ij} dx^{idx^j} \qquad (i, j = 1, \dots, n-1)$$

have nonpositive curvature, i.e., have nonpositive curvature at each point in each two-dimensional direction. In order that the $n$-dimensional Riemannian space $V^n$ with fundamental form

$$ds^2 = g_{ij} dx^{idx^j} + (dx^n)^2, \qquad g_{ij} = [\lambda(x^n)]^{1/(n-1)} a_{ij}$$

have nonpositive curvature, it is sufficient that condition $(**)$ be satisfied.

**2°.** We give some special cases of the theorem. Consider an arbitrary $(n-1)$-dimensional Riemannian space $V^{n-1} \in I_{n-1}$. Let

$$(ds')^2 = g_{ij} dx^{idx^j} \qquad (i, j = 1, \dots, n-1)$$

its fundamental form in some system of coordinates. In the direct product $V^n = V^{n-1} \times R$ of the space $V^n$ with the real line $R$, introduce a metric by means of the line element

$$ds^2 = a(x^n) g_{ij} dx^i dx^j + (dx^n)^2,$$

where $a$ is a positive twice continuously differentiable function of one real variable. It follows easily from the theorem that if $(\sqrt{a})'' \geq 0$, then $V^n \in I_n$.

Let us note that if $V^{n-1}$ is Lobachevskii space of curvature $K$, then, as is not hard to see, $V^n$ for $a = \operatorname{ch}^2 \sqrt{-K}\, x$ is also Lobachevskii space of curvature $K$. Thus every Lobachevskii space is a space with the Euclidean isoperimetric inequality.

Consider a Riemannian space $V^n$ admitting a system of $n$-orthogonal hypersurfaces, i.e., a space whose fundamental form in some system of coordinates has the form

$$ds^2 = g_{11}(dx^1)^2 + \dots + g_{nn}(dx^n)^2.$$

With the aid of our theorem it is not difficult to establish that, in order that the space $V^n \in I_n$, it is sufficient that the following conditions be satisfied:

1) $g_{nn} = \text{const}$, $g_{ii} = \alpha_{i\,i+1}(x^{i+1}) \cdots \alpha_{in}(x^n)$, where $\alpha_{ij}$ is a positive twice continuously differentiable function of one real variable;

2) for each $i = 2, \dots, n$ the inequality

$$\left[(\alpha_{1i} \cdots \alpha_{i-1\,i})^{1/(2i-2)}\right]'' \geq 0$$

holds.

We shall give precise definitions of the symmetrizations by means of which the theorem is proved.

3°. The symmetrization $S_1$ assigns to an arbitrary body $Q$ of the space $V^n$ a certain body $S_1 Q$ of an $n$-dimensional Riemannian space $W^n$ with fundamental form

$$ds^2 = [\lambda(y^n)]^{1/(n-1)}[(dy^1)^2 + \cdots + (dy^{n-1})^2] + (dy^n)^2.$$

Denote by $\Gamma$ the geodesic $y^1 = \cdots = y^{n-1} = 0$. Let the body $S_1 Q$ be determined by the following conditions:

1) $\Gamma$ is its axis of rotation;

2) for every real $x$, the area of the section of the body $Q$ by the hypersurface $x^n = x$ is equal to the area of the section of the body $S_1 Q$ by the hypersurface $y^n = x$.

The symmetrization $S_1$, in the case when the space $V^n$ is Euclidean, coincides with the well-known Schwarz symmetrization ([3]).

4°. The symmetrization $S_2$ assigns to an arbitrary body $Q$ of the space $W^n$ a certain body $S_2 Q$ of Euclidean space $E^n$. Denote by $\Gamma_\rho$ the set of points in the space $W^n$ the distance from each of which to the geodesic $\Gamma$ is equal to $\rho$. In $E^n$ introduce some rectangular coordinate system $(z^1, \dots, z^n)$. Denote by $\gamma$ the line $z^1 = \cdots = z^{n-1} = 0$, and by $\gamma_\rho$ the set of points at distance $\rho$ from $\gamma$. Construct in $E^n$ a body $\bar{Q}$, determined by the following conditions:

1) $\bar{Q}$ is symmetric with respect to the hyperplane $z^n = 0$;

2) for every $\rho > 0$, the area of the section of the body $Q$ by the hypersurface $\Gamma_\rho$ is equal to the area of the section of the body $\bar{Q}$ by the hypersurface $\gamma_\rho$.

Replace each section of the body $\bar{Q}$ by the hypersurface $z^n = \text{const}$ by its convex hull. The body obtained in this way will be denoted by $S_2 Q$.

5°. It is known ([3]) that Schwarz symmetrization does not decrease the volume of a body and does not increase the area of the boundary of this body. It can be proved (specifically for this purpose the conditions ($*$) and ($**$) were imposed on the space $V^n$) that the symmetrization $S_1$ and the superposition $S_2 S_1$ of the symmetrizations $S_1$ and $S_2$ possess this same property. Thus, for an arbitrary body $Q \subset V^n$ we have succeeded in constructing a body $S_2 S_1 Q \subset E^n$ with no smaller volume and no greater boundary area, and this means that $V^n \in I_n$.

Institute of Mathematics
Siberian Branch of the Academy of Sciences of the USSR
Moscow

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## CITED LITERATURE

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*Note: Figure translations are in progress. See original paper for figures.*