ASYMPTOTICS OF THE SOLUTION OF A PROBLEM WITH AN INITIAL JUMP FOR NONLINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS WITH A SMALL PARAMETER AT THE HIGHEST DERIVATIVE

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ASYMPTOTICS OF THE SOLUTION OF A PROBLEM WITH AN INITIAL JUMP FOR NONLINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS WITH A SMALL PARAMETER AT THE HIGHEST DERIVATIVE

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In the present paper we shall consider a nonlinear system of ordinary differential equations with a small parameter, which is the direct analogue, for a single scalar equation, of the form*

$$\varepsilon\, d^s y/dt^s = f(t, y, dy/dt, \dots, d^{s-1}y/dt^{s-1}).$$

§ 1. Suppose we have the nonlinear system:

$$\varepsilon\, dz/dt = F(z, y, x, t), \qquad dy/dt = G(z, y, x, t), \qquad dx/dt = H(z, y, x, t). \quad (1)$$

with initial conditions

$$x(t, \varepsilon)\,\big|_{t=0} = x_0^0, \qquad y(t, \varepsilon)\,\big|_{t=0} = y_0^0, \qquad z(t, \varepsilon)\,\big|_{t=0} = z_0^0(\varepsilon), \qquad (2)$$

where $\varepsilon > 0$ is a small parameter; x_0^0 and y_0^0 are certain constants independent of ε; $z_0^0(\varepsilon) \to \infty$ as $\varepsilon \to 0$. Assume that the right-hand sides F, G, and H of system (1) grow as $z \to \infty$ like z^n, z^m, and z^k, respectively, and admit the representations

$$F \equiv z^n \left[f_0(t, x, y) + \sum_{i>0} f_i(t, x, y) z^{-i} \right],$$

$$G \equiv z^m \left[g_0(t, x, y) + \sum_{i>0} g_i(t, x, y) z^{-i} \right], \qquad (3)$$

$$H \equiv z^k \left[h_0(t,x,y) + \sum_{i>0} h_i(t,x,y) z^{-i} \right],$$

$$f_0(t,x,y) < 0, \qquad g_0(t,x,y) > 0, \tag{4}$$

where n, m, and k are certain real numbers satisfying the conditions:
a) $k < m = n,\ n \geq 1$;
b) $n-1 < m < n,\ k < m$;
c) $k < m = n-1,\ n > 1$.

Then, with the corresponding choice of the singularity of the function $z_0^0(\varepsilon)$, as $\varepsilon \to 0$ the solution $z(t,\varepsilon)$, $y(t,\varepsilon)$, $x(t,\varepsilon)$ of problem (1) and (2) will tend to the solution of the degenerate system of equations corresponding to system (1):

$$0 = F(\bar{z},\bar{y},\bar{x},t), \qquad d\bar{y}/dt = G(\bar{z},\bar{y},\bar{x},t), \qquad d\bar{x}/dt = H(\bar{z},\bar{y},\bar{x},t), \tag{5}$$

where $\bar{y}(t)$ will no longer satisfy the previous initial condition $\bar{y}(0) = y_0^0$, but will satisfy another initial condition, in contrast to the variable $\bar{x}(t)$, for which the previous condition holds:

$$\bar{x}(t)\,\Big|_{t=0} = x_0^0, \qquad \bar{y}(t)\,\Big|_{t=0} = y_0^0 + \Delta y_0^0, \tag{6}$$

where the quantity Δy_0^0 will be called the initial jump of the function y. The existence of a finite initial jump of the function y depends on the order of growth of the func-

* If, for example, in (1) $k = 0$ and the variable x is an $(s-2)$-dimensional vector, while y and z are scalars, then system (1) is an analogue of a differential equation of order s.

the function $z_0^0(\varepsilon)$ as $\varepsilon \to 0$, and the order of growth of $z_0^0(\varepsilon)$, in turn, depends essentially on the character of the nonlinearity of the right-hand side of the system of equations (1), namely:

$$z_0^0(\varepsilon) = \begin{cases} z_0^0/\varepsilon, & m = n,\ n \geqslant 1, \\ z_0^0/\varepsilon^{1/(m+1-n)}, & n-1 < m < n,\ n \geqslant 1, \\ z_0^0 e^{c/\varepsilon}, & m = n-1,\ n > 1,\ c > 0, \end{cases} \tag{7a}$$

(7b)

(7c)

where z_0^0 is some constant, and, for definiteness, we shall assume $z_0^0 > 0$. (If $z_0^0 < 0$, then in (3) z must be replaced by $-z$.)

In the present note the case (7a) is investigated; the remaining cases (7b) and (7c) can be studied in a completely analogous way. To determine the magnitude of the initial jump Δy_0^0 we have the equation

$$z_0^0 = -\int_{y_0^0}^{y_0^0+\Delta y_0^0} \frac{f_0(0, x_0^0, y)}{g_0(0, x_0^0, y)}\, dy. \tag{8}$$

To construct the asymptotics of the solution of the problem (1), (2), where $z_0^0(\varepsilon)$ is expressed by (7a), we divide the interval $[0, T]$ (see (1, 2)) into three zones. In each of the three zones obtained, the system of equations (1) is solved in a different way. In the first zone $0 \leqslant t \leqslant t_1^0$, where $t_1^0 = O(\varepsilon^{n(1-\sigma)})$, $n > 1$, the variable quantity z changes from a quantity of order $O(1/\varepsilon)$ to a quantity of order $O(1/\varepsilon^{1-\sigma})$, the variable quantity y changes from y_0^0 to $\bar{y}_0^0 + O(\varepsilon^{\sigma})$, where $\bar{y}_0^0 \equiv y_0^0 + \Delta y_0^0$, and x changes from x_0^0 to $x_0^0 + O(\varepsilon^{1-\sigma} + \varepsilon^{(n-1)(1-\sigma)} + \varepsilon^{\sigma+(n-k)(1-\sigma)})$. In the second zone $t_1^0 \leqslant t \leqslant t_2^0$, where $t_2^0 = O(\varepsilon)$, the variable quantity z changes from a quantity of order $O(1/\varepsilon^{1-\sigma})$ to some finite quantity (as $\varepsilon \to 0$), and the jump for y is small together with ε. In the third zone $t_2^0 \leqslant t \leqslant T$, the solution of the problem (1) and (2) rapidly approaches the solution of the degenerate problem (5), (6) and subsequently remains in a small neighborhood of the solution of the degenerate problem. Our goal now is to construct the asymptotics only in the first two zones. In the third zone the asymptotics can be constructed by the method of A. B. Vasil' eva (3).

§ 2. In the first zone we make the change of variables:

$$t = \varepsilon^n \tau, \qquad \varepsilon z = u, \qquad n > 1 \tag{9}$$

and the zone $0 \leqslant t \leqslant t_1^0$ passes into the zone $0 \leqslant \tau \leqslant \tau_1^0$, where $\tau_1^0 = O(1/\varepsilon^{(n-1)\sigma})$. Then the initial problem (1) and (2) is transformed into the following one:

$$\begin{aligned}
\frac{du}{d\tau} &= u^n \left[f_0(\varepsilon^n \tau, x, y) + \sum_{i>0} \varepsilon^i f_i(\varepsilon^n \tau, x, y) u^{-i} \right], \\
\frac{dy}{d\tau} &= u^n \left[g_0(\varepsilon^n \tau, x, y) + \sum_{i>0} \varepsilon^i g_i(\varepsilon^n \tau, x, y) u^{-i} \right], \\
\frac{dx}{d\tau} &= \varepsilon^{n-k} u^k \left[h_0(\varepsilon^n \tau, x, y) + \sum_{i>0} \varepsilon^i h_i(\varepsilon^n \tau, x, y) u^{-i} \right],
\end{aligned} \tag{10}$$

$$u(\tau, \varepsilon)\big|_{\tau=0} = z_0^0, \qquad y(\tau, \varepsilon)\big|_{\tau=0} = y_0^0, \qquad x(\tau, \varepsilon)\big|_{\tau=0} = x_0^0. \tag{11}$$

We seek an approximate solution of the problem (10), (11) in the form:

$$
\begin{aligned}
u(\tau,\varepsilon) &= u_0(\tau) + {\sum_{s,p,q}}' \varepsilon^{sn+p+q(n-k)} u_{spq}(\tau),\\
y(\tau,\varepsilon) &= y_0(\tau) + {\sum_{s,p,q}}' \varepsilon^{sn+p+q(n-k)} y_{spq}(\tau),\\
x(\tau,\varepsilon) &= x_0(\tau) + {\sum_{s,p,q}}' \varepsilon^{sn+p+q(n-k)} x_{spq}(\tau).
\end{aligned}
\tag{12}
$$

Substituting (12) into (10) and (11) and equating coefficients of like powers of ε, we obtain a sequence of differential equations for $u_{spq}(\tau)$, $y_{spq}(\tau)$, $x_{spq}(\tau)$:

$$
\begin{aligned}
du_0/d\tau &= f_0(y_0, x_0^0, 0)u_0^n, \qquad u_0(0) = z_0^0,\\
dy_0/d\tau &= g_0(y_0, x_0^0, 0)u_0^n, \qquad y_0(0) = y_0^0,
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
du_{spq}/d\tau &= nf_0(y_0, x_0^0, 0)u_0^{n-1}u_{spq} + u_0^n f_{0y}(y_0, x_0^0, 0)y_{spq} + \Phi_{spq}^{(1)},\\
dy_{spq}/d\tau &= ng_0(y_0, x_0^0, 0)u_0^{n-1}u_{spq} + u_0^n g_{0y}(y_0, x_0^0, 0)y_{spq} + \Phi_{spq}^{(2)},\\
dx_{spq}/d\tau &= \Phi_{spq}^{(3)}, \qquad u_{spq}(0) = y_{spq}(0) = x_{spq}(0) = 0,
\end{aligned}
\tag{14}
$$

where $\Phi_{spq}^{(1)}$ and $\Phi_{spq}^{(2)}$ are known functions of u_{ijl}, y_{ijl}, x_{ijl} and x_{spq}, while the function $\Phi_{spq}^{(3)}$ is expressed through u_{ijl}, y_{ijl}, x_{ijl}, $i+j+l < s+p+q$.

For $u_0(\tau)$, $y_0(\tau)$, and $x_0(\tau)$ the estimates

$$
u_0(\tau) = O((1+\tau)^{-1/(n-1)}), \qquad \bar{y}_0^0 - y_0(\tau) = O((1+\tau)^{-1/(n-1)}),
$$

$$
x_0(\tau) = x_0^0, \qquad 0 \le \tau \le \tau_1^0, \tag{15}
$$

hold, and for $u_{spq}(\tau)$, $y_{spq}(\tau)$, and $x_{spq}(\tau)$ the estimates

$$
(u,y)_{spq} = O\left((1+\tau)^{[s(n-1)+p-1+q(n-k-1)]/(n-1)}\right),
$$

$$
x_{spq} = \begin{cases} O\left((1+\tau)^{[s(n-1)+p-1+q(n-k-1)]/(n-1)}\right), & q > 0,\\ 0, & q = 0. \end{cases} \tag{16}
$$

Theorem 1. Every solution u, y, and x of problem (10) and (11) in the first zone $0 \le \tau \le \tau_1^0$ has the asymptotic expansion

$$u(\tau,\varepsilon)=u_0(\tau)+\sum_{s+p+q=1}^{N}\varepsilon^{sn+p+q(n-k)}u_{spq}(\tau)+R_N^{(1)},$$

$$y(\tau,\varepsilon)=y_0(\tau)=\sum_{s+p+q=1}^{N}\varepsilon^{sn+p+q(n-k)}y_{spq}(\tau)+R_N^{(2)}, \tag{17}$$

$$x(\tau,\varepsilon)=x_0(\tau)+\sum_{s+p+q=1}^{N}\varepsilon^{sn+p+q(n-k)}x_{spq}(\tau)+R_N^{(3)},$$

where

$$(R_N^{(1)},R_N^{(2)})=O\left(\sum_{s+p+q=N+1}\varepsilon^{\sigma(1+q+s)+(sn+p+q(n-k))(1-\sigma)}\right),$$

$$R_N^{(3)}=O\left(\sum_{s+p+q=N+1}\varepsilon^{\sigma(q+s)+(sn+p+q(n-k))(1-\sigma)}\right). \tag{18}$$

It can be verified directly that the value $t=t_1^0$ corresponds to

$$z(t_1^0,\varepsilon)=z_1^0(\varepsilon)=O(1/\varepsilon^{1-\sigma}),\qquad 0<\sigma<1,$$

$$y(t_1^0,\varepsilon)=y_1^0(\varepsilon)=\bar{y}_0^0+O(\varepsilon^{\sigma}), \tag{19}$$

$$x(t_1^0,\varepsilon)=x_1^0(\varepsilon)=x_0^0+O(\varepsilon^{(n-1)(1-\sigma)}+\varepsilon^{1-\sigma}+\varepsilon^{\sigma+(n-k)(1-\sigma)}).$$

§ 3. In the second zone the system of differential equations (1) is solved under the initial conditions (19). The change of variable $\tau=(t-t_1^0)/\varepsilon$ reduces problem (1) and (19) to the problem:

$$\begin{aligned} dz/d\tau&=F(z,y,x,t_1^0+\varepsilon\tau), & z(\tau,\varepsilon)\big|_{\tau=0}&=z_1^0(\varepsilon),\\ dy/d\tau&=\varepsilon G(z,y,x,t_1^0+\varepsilon\tau), & y(\tau,\varepsilon)\big|_{\tau=0}&=y_1^0(\varepsilon),\\ dx/d\tau&=\varepsilon H(z,y,x,t_1^0+\varepsilon\tau), & x(\tau,\varepsilon)\big|_{\tau=0}&=x_1^0(\varepsilon). \end{aligned} \tag{20}$$

The zone $t_1^0\le t\le t_2^0$ passes into the zone $0\le\tau\le\tau_2^0$, where τ_2^0 is a sufficiently small number, but fixed as $\varepsilon\to 0$.

We seek the formal solution of problem (20) in the form of a series in ε with coefficients depending not only on τ, but also on ε:

$$\begin{aligned} z(\tau,\varepsilon) &= z_0(\tau,\varepsilon)+\varepsilon z_1(\tau,\varepsilon)+\varepsilon^2 z_2(\tau,\varepsilon)+\cdots,\\ y(\tau,\varepsilon) &= y_0(\tau,\varepsilon)+\varepsilon y_1(\tau,\varepsilon)+\varepsilon^2 y_2(\tau,\varepsilon)+\cdots,\\ x(\tau,\varepsilon) &= x_0(\tau,\varepsilon)+\varepsilon x_1(\tau,\varepsilon)+\varepsilon^2 x_2(\tau,\varepsilon)+\cdots. \end{aligned} \tag{21}$$

In the first approximation we obtain

$$dz_0/d\tau = F_0(z_0, y_1^0(\varepsilon), x_1^0(\varepsilon), t_1^0(\varepsilon)), \qquad z_0(0,\varepsilon)=z_1^0(\varepsilon).$$

Hence, taking into account $F_0 = O(z_0^n)$, we obtain the estimate for $z_0(\tau,\varepsilon)$:

$$z_0(\tau,\varepsilon) = O\left[\frac{1}{\left((1/z_1^0(\varepsilon))^{n-1}+\tau\right)^{1/(n-1)}}\right]. \tag{22}$$

It follows directly from estimate (22) that $z_0(\tau_2^0,\varepsilon)$ has a finite limit as $\varepsilon\to 0$.

Substituting now (21) into (20) and equating the coefficients of identical powers of ε, we obtain a sequence of differential equations for $z_p(\tau,\varepsilon)$, $y_p(\tau,\varepsilon)$, and $x_p(\tau,\varepsilon)$, $p>0$:

$$\begin{aligned} \frac{dz_p}{d\tau} &= \frac{\partial F_0}{\partial z} z_p + \Phi_p^{(1)}(\tau), \qquad z_p(0,\varepsilon)=0,\\ dy_p/d\tau &= \Phi_p^{(2)}(\tau), \qquad y_p(0,\varepsilon)=0,\\ dx_p/d\tau &= \Phi_p^{(3)}(\tau), \qquad x_p(0,\varepsilon)=0, \end{aligned} \tag{23}$$

where $\Phi_p^{(i)}(\tau)$ are already known functions of z_s, y_s, and x_s, $s<p$.

The estimates

$$z_p(\tau,\varepsilon) = O(z_0^p(0,\varepsilon) z_0(\tau,\varepsilon)), \qquad y_p(\tau,\varepsilon) = O(z_0^p(0,\varepsilon)),$$

$$x_p(\tau,\varepsilon) = O\left(\sum_{s=1}^{p} z_0^{p-s}(0,\varepsilon)\left(\frac{z_0^s(\tau,\varepsilon)}{z_0^{s(n-k)}(\tau,\varepsilon)} + \frac{z_0^s(0,\varepsilon)}{z_0^{s(n-k)}(0,\varepsilon)}\right)\right). \tag{24}$$

hold.

In this zone the following is valid.

Theorem 2. *Every solution of problem (20) in the second zone $0\le\tau\le\tau_2^0$ has the asymptotic expansion*

$$z(\tau,\varepsilon)=\sum_{p=0}^{N}\varepsilon^p z_p(\tau,\varepsilon)+R_N^{(1)}(\tau,\varepsilon),\qquad y(\tau,\varepsilon)=\sum_{p=0}^{N}\varepsilon^p y_p(\tau,\varepsilon)+R_N^{(2)}(\tau,\varepsilon),$$

$$x(\tau,\varepsilon)=\sum_{p=0}^{N}\varepsilon^p x_p(\tau,\varepsilon)+R_N^{(3)}(\tau,\varepsilon), \tag{25}$$

where

$$R_N^{(1)}=O(\varepsilon^{(N+1)\sigma-(1-\sigma)}),\qquad R_N^{(2)}=O(\varepsilon^{(N+1)\sigma}),$$

$$R_N^{(3)}=O\left(\sum_{s=1}^{N+1}\varepsilon^{(N+1)\sigma+s(n-k)(1-\sigma)}\right). \tag{26}$$

Remark. The method set forth above carries over directly to the case when the variables y and x are vector quantities.

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Note: Figure translations are in progress. See original paper for figures.