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Abstract

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V. Ya. RIVKIND, N. N. URAL' TSEVA

A PRIORI ESTIMATES FOR QUASILINEAR PARABOLIC EQUATIONS WITH DISCONTINUOUS COEFFICIENTS AND THEIR APPLICATION IN APPROXIMATE METHODS

(Presented by Academician V. I. Smirnov on 16 VII 1968)

In many works, a priori estimates have been derived for solutions of linear parabolic equations and systems with discontinuous coefficients (see, for example, ([1–4]), etc.). These estimates make it possible to prove the existence of generalized solutions of diffraction problems ([1,2]) and to investigate their smoothness. The solvability of such problems in the classical sense was proved for the case of general systems of parabolic type by methods of potential theory in ([5]). The case of one spatial variable has been well studied (see ([6,7]), etc.).

In the present work, a priori estimates are derived for solutions of diffraction problems for quasilinear parabolic equations, and the classical solvability of these problems is proved. In addition, it is shown here how to use the estimates obtained for finding approximate solutions, and energy estimates of the rate of convergence are derived for various approximate methods. All notation in the paper coincides with ([2,3,8]).

Let, in a bounded cylindrical domain $Q = \Omega \times [0, T]$, $\Omega = \sum_{i=1}^{l} \Omega_i$, a boundary-value problem be considered for the equation

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x_i} a_{ij}(x, t, u) \frac{\partial u}{\partial x_j} + b\left(x, t, u, \frac{\partial u}{\partial x}\right) = 0, \tag{1}$$

where $a_{ij}(x, t, u)$ have discontinuities with respect to x on a finite number of smooth surfaces $\Gamma' = \Gamma \times [0, T]$ (Γ are the interfaces of the Ω_i), on which the nonlinear matching conditions are imposed:

$$\left[a_{ij}(x, t, u) \frac{\partial u}{\partial x_j} \cos(\mathbf{n}, x_i)\right]_{x\in\Gamma'} = 0; \qquad [u]_{x\in\Gamma'} = 0. \tag{2}$$

Here $[v]$ denotes the jump of the function v when passing through the surface Γ; $\mathbf{n}$ is the normal to Γ. On the lateral surface $S' = S \times [0,T]$ of the cylinder $\overline{Q} = \overline{\Omega} \times [0,T]$ and for $t = 0$, the solution satisfies the conditions

$$u|_{x\in S'} = 0; \qquad u|_{t=0} = 0. \tag{3}$$

By a classical solution of problem (1)–(3) we shall mean a function

$$u(x,t) \in C^{2,1}(Q_i) \cap H^{1+\alpha,(1+\alpha)/2}(\overline{Q}_i) \cap H^{\alpha,\alpha/2}(\overline{Q});$$

$$Q_i = \Omega_i \times [0,T]; \qquad i = 1,\dots,l; \quad d > 0,$$

satisfying equation (1) and conditions (2), (3), and such that

$$\sum_{i=1}^{l} \int_{Q_i} \left[\left(\frac{\partial u}{\partial t}\right)^2 + \sum_{k,m=1}^{n} \left(\frac{\partial^2 u}{\partial x_k \partial x_m}\right)^2 \right] dx\, dt < \infty.$$

Theorem 1. *Let, for $(x,t) \in \overline{Q}$, $|u| \leq M$ and arbitrary p, the functions $a_{ij}(x,t,u)$, $b(x,t,u,p)$ be continuous in the arguments u, p, have dis-*

discontinuities of the first kind in x when crossing Γ' and satisfy the conditions:

$$\nu \sum_{i=1}^{n} \xi_i^2 \leqslant a_{ij}\xi_i\xi_j \leqslant \mu \sum_{i=1}^{n} \xi_i^2; \tag{4}$$

$$\max_Q |b(x,t,u,p)| \leqslant \mu(1+p^2); \qquad |p| = \left(\sum_{k=1}^{n} p_k^2\right)^{1/2}; \tag{5}$$

$$\max_Q \left(\left|\frac{\partial b}{\partial p}\right| (1+|p|) + \left|\frac{\partial b}{\partial u}\right| + \left|\frac{\partial b}{\partial t}\right| \right) \leqslant \mu(1+p^2); \tag{6}$$

$$\sum_{k;m=1}^{n} \left[\max_Q \left|\frac{\partial a_{km}}{\partial u}\right| + \max_{Q_i} \left\{ \left|\frac{\partial a_{km}}{\partial x}\right| + \left|\frac{\partial a_{km}}{\partial t}\right| \right\} \right] \leqslant \mu; \tag{7}$$

$\mu = \text{const} > 0$.

Suppose that the different components of the surfaces Γ' and S' do not intersect and belong to the class O^2. Then for every classical solution $u(x,t)$ of problem (1)–(3) the estimate

$$\sum_{i=1}^{l} \left| \frac{\partial u}{\partial x} \right|_{Q_i}^{(\alpha)} \leqslant M_1, \tag{8}$$

holds, where the constants M_1 and $\alpha \in (0,1)$ depend only on ν and μ from (4)–(7) and on the properties of the surfaces S' and Γ'.

Estimate (8) is obtained by the methods of [2] and is of a local character. By known methods one proves

Lemma 1. Let, for the functions $a_{ij}(x,t,u)$, $b(x,t,u,p)$, for arbitrary u, p, the conditions

$$0 \leqslant a_{ij}(x,t,u)\xi_i\xi_j \leqslant \mu \sum_{i=1}^{n} \xi_i^2; \qquad x \in \Omega; \quad t \in [0,T]; \tag{9}$$

$$-ub(x,t,u,p) \leqslant C_0 p^2 + C_1 u^2 + C_2; \qquad x \in \Omega; \quad t \in [0,T]; \tag{10}$$

$$\nu_1 \sum_{i=1}^{n} \xi_i^2 \leqslant a_{ij}\xi_i\xi_j; \qquad (x,t) \in \Gamma', \tag{11}$$

hold, where ν_1, μ, C_i $(i = 0,1,2)$ are positive constants.

Then, for any classical solution of problem (1)–(3), the inequality

$$\max_{\overline{Q}} |u(x,t)| \leqslant M, \tag{12}$$

is valid, where the constant M is determined only by the quantities $\nu_1, \mu, C_0, C_1, C_2$ from (9)–(11) and by the properties of the surface Γ'.

Under other conditions an estimate of the maximum of the modulus of the solution of problem (1)–(3) is derived with the aid of integral inequalities (see, for example, Theorem 1, Ch. 5 [2]).

From Theorem 1, Lemma 1, and the existence theorems for linear parabolic equations with discontinuous coefficients proved in [2] and [5], on the basis of the Leray–Schauder theorem one derives

Theorem 2. Suppose that the functions $a_{ij}(x,t,u)$, $b(x,t,u,p)$ satisfy the conditions of Lemma 1 and Theorem 1 with the constant M from (12); for $(x,t) \in Q_i$, $|u| \leqslant M$, $|p| \leqslant M_1$, where M and M_1 are the constants from inequalities (12), (8), the functions $b(x,t,u,p)$, $\partial a_{ij}(x,t,u)/\partial u$, $\partial a_{ij}/\partial x_k$, $\partial a_{ij}(x,t,u)/\partial t$ are Hölder-continuous in the variables x, u with exponent α, and in t with exponent $\alpha/2$; $S \in H^{2+\alpha}$, $b(x,0,0,0) = 0$.

Then problem (1)–(3) has a classical solution.

Thus, the classical solvability of the stated problem has been established. We shall turn to the search for approximate solutions. Consider the equa-

equation

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x_i}\hat{a}_{ij}(x,t,u)\frac{\partial u}{\partial x_j} + \hat{b}\left(x,t,u,\frac{\partial u}{\partial x}\right) = 0, \tag{13}$$

where the functions $\hat{a}_{ij}$, $\hat{b}$ are defined by the equalities

$$\hat{a}_{ij}(x,t,u) = a_{ij}(x,t,u)\xi(u,0) + a_{ij}(x,t,0)(1-\xi(u,0)); \tag{14}$$

$$\hat{b}(x,t,u,p) = b(x,t,u,p)\xi(u,p) + b(x,t,u,p)(1-\xi(u,p)), \tag{15}$$

where $\xi(u,p)$ is a smooth nonnegative function in R_{n+1}, equal to 1 for $|u| \leq M$, $|p| \leq M_1$, and equal to zero for $|u| \geq M+1$, $|p| \geq M_1+1$, where M, M_1 are determined by inequalities (8), (12).

From Theorem 1 it is clear that every classical solution of problem (1), (2), (3) is a solution of problem (13), (2), (3) with bounded nonlinearities. The latter can be solved by various approximate methods, the convergence-rate estimate for which is given below.

We assume below that the conditions of Theorem 2 are satisfied. For the solution of the problem with bounded nonlinearities, consider the linear difference scheme at each step

$$t_{j+1} = \sum_{k=1}^{j+1}(\Delta t)_k$$

(see [1,3,8,9])

$$u^h_{\bar{t}}(x^h,t_{j+1}) - \left(\hat{a}^h_{im}(x^h,t_{j+1},u^h(x^h,t_j))u^h_{x_i}(x^h,t_{j+1})\right)_{\bar{x}_m} +$$

$$+\hat{b}^h\left(x^h,t_{j+1},u^h(x^h,t_j),u^h_x(x^h,t_j)\right) = 0; \tag{16}$$

$$u^h\big|_{x\in S'_h} = 0; \qquad u^h\big|_{t=0} = 0. \tag{17}$$

For the solution u^h of (16)–(17), just as in (8), one proves

Theorem 3. *The solution of problem (16)–(17) for finite h exists, is unique, and tends to the classical solution of problem (1)–(3) in such a way that*

$$\max_{t\in[0,T]} \|u - u^{h'}\|_{L_2(\Omega)} + \left\|\frac{\partial u}{\partial x} - \frac{\partial u^{h'}}{\partial x}\right\|_{L_2(Q)} \leq C\left((\overline{\Delta t})^{\alpha} + h^{1/2}\right), \tag{18}$$

where the constants C, α depend on the properties of the functions $a_{ij}(x,t,u)$, $b(x,t,u,p)$ and of the boundaries S' and Γ',

$$(\overline{\Delta t})^{\alpha} = \left[\frac{1}{t_{j+1}}\sum_{k=0}^{j+1}(\Delta t)_k^{2\alpha+1}\right]^{1/2};$$

$u^{h'}$ is the multilinear interpolation of u^h.

Let us note that if one abandons the requirement of homogeneity of the difference scheme near the surfaces S' and Γ', then, as in [10,11], one can obtain estimates of the type

$$\max_{t\in[0,T]} \|u - u_{\Delta}\|_{L_2(\Omega)} + \left\|\frac{\partial u}{\partial x} - \frac{\partial u_{\Delta}}{\partial x}\right\|_{L_2(Q)} \leq C\left(h + (\overline{\Delta t})^{\alpha}\right), \tag{19}$$

where u_{Δ} is the simplicial interpolation of u^h.

The linear algebraic systems obtained on each layer t_{j+1} from the difference scheme (16), (17), can be solved by ordinary iterative or direct methods. We also give estimates for the Rothe and Galerkin methods. If the solution is sought in the form

$$u_t^*(x,t_{j+1}) - \frac{\partial}{\partial x_i}\left(\hat{a}_{im}(x,t_{j+1},u^*(x,t_j))\frac{\partial u^*}{\partial x_m}\right) +$$
$$+\hat{b}\left(x,t_{j+1},u^*(x,t_j),\frac{\partial u^*}{\partial x}(x,t_j)\right) = 0, \tag{20}$$

where u^* satisfies conditions (2), (3), then as $(\Delta t)_k \to 0,\ k = 1,\ldots,\overline{\Delta t}$

the solution (20) tends to the exact solution $u(x,t)$ in such a way that

$$\max_{t\in[0,T]} \|u^* - u\|_{L_2(\Omega)} + \left\|\frac{\partial u}{\partial x} - \frac{\partial u^*}{\partial x}\right\|_{L_2(Q)} \leqslant C((\overline{\Delta t})^{\alpha}), \tag{21}$$

where C depends on the same quantities as in inequality (19). For the approximate solution obtained by the Galerkin method (see ([3]), Ch. 5), the convergence-rate estimate has the form

$$\max_{t\in[0,T]} \|u^N - u\|_{L_2(\Omega)} + \left\|\frac{\partial u^N}{\partial x} - \frac{\partial u}{\partial x}\right\|_{L_2(Q)} \leqslant$$

$$\leqslant \max_{t\in[0,T]} \|u - P_{Nu}\|_{L_2(\Omega)} + \left\|\frac{\partial u}{\partial x} - \frac{\partial P_{Nu}}{\partial x}\right\|_{L_2(Q)}, \tag{22}$$

where C depends on the same quantities as in inequality (18), and P_{Nu} is the projection of the exact solution onto the subspace $W_2^1(\Omega)$, composed of the coordinate functions $\psi_i(x)$, $i = 1, \ldots, N$. For each concrete set of functions $\psi_i(x)$, it is easy to obtain an estimate of the right-hand side of (22) from Theorem 1 and thereby the required effective estimate. The nonlinear system of ordinary differential equations arising in the Galerkin method can be solved by the Euler, Adams, Runge-Kutta methods, etc.

Similar convergence-rate estimates are easily derived, using the a priori estimates established above, for other methods as well: the method of integral relations, the method of lines, and so on.

The results obtained in the paper generalize to the case of systems of equations of the form:

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x_i}\left(a_{ij}(x,t,u)\frac{\partial u}{\partial x_j}\right) + B_i(x,t,u)\frac{\partial u}{\partial x_i} + C(x,t,u) = 0; \tag{23}$$

$$\left[a_{ij}(x,t,u)\frac{\partial u}{\partial x_j}\cos(\mathbf{n}, x_i)\right]\Bigg|_{x\in\Gamma'} = 0; \qquad [u]_{x\in\Gamma'} = 0; \tag{24}$$

$$u\big|_S = \varphi(t); \qquad u\big|_{t=0} = \psi(x).$$

(Equalities (23)–(24) are understood componentwise.) B, C are matrices whose elements satisfy the conditions of Theorems 1, 2, certain additional compatibility conditions at $t = 0$, and also certain cases of other boundary conditions.

Let us also note that the approximate solution of these parabolic equations can sometimes be regarded as a stabilization problem for the solution. In this case such boundary-value problems can be regarded as an iterative process for the corresponding elliptic equation. The a priori estimates required for this and the classical solvability of boundary-value problems for elliptic equations are established in (12).

Leningrad State University
named after A. A. Zhdanov

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Note: Figure translations are in progress. See original paper for figures.