AN INVESTIGATION OF THE HERMITE-FEJÉR INTERPOLATION PROCESS

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Abstract

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MATHEMATICS

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AN INVESTIGATION OF THE HERMITE-FEJÉR INTERPOLATION PROCESS

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1°. Let a matrix of numbers be given

$$-1 \leq x_n^{(n)} < x_{n-1}^{(n)} < \dots < x_1^{(n)} \leq 1, \qquad n = 1, 2, \dots \tag{m}$$

Denote by C the set of all functions $f(x)$ continuous on $[-1, 1]$. By $L_n(f, x)$ we shall denote the Lagrange interpolation polynomial of degree $(n-1)$, constructed for the function $f(x)$ and the n-th row of the matrix (m).

According to the classical theorem of S. N. Bernstein–G. Fejér, there is no matrix of nodes (m) for which, for every $f \in C$, the relation $L_n(f, x) \to f(x)$, $n \to \infty$, holds uniformly on $[-1, 1]$.

In this connection the theorem of L. Fejér is of interest; it is formulated as follows. Let the matrix (m) be composed of the roots of the polynomials of P. L. Chebyshev $T_n(x) = \cos n \arccos x$, i.e.

$$x_k^{(n)} = \cos \frac{2k-1}{2n}\pi, \qquad k = 1, 2, \dots, n; \ n = 1, 2, \dots, \tag{1}$$

and let $H_n(f) \equiv H_n(f, x)$ be the polynomial of degree $(2n - 1)$, constructed for $f \in C$ and the nodes (1), and uniquely determined by the conditions $H_n(f, x_k^{(n)}) = f(x_k^{(n)})$, $H'_n(f, x_k^{(n)}) = 0$, $k = 1, 2, \dots, n$. Then for every $f \in C$ the relation $H_n(f, x) \to f(x)$, $n \to \infty$, holds uniformly on $[-1, 1]$.

The interpolation process $\{H_n(f)\}_{n=1}^{\infty}$ is usually called the Hermite-Fejér interpolation process. It was the subject of important investigations by L. Fejér and his students.

2°. In (1) it was found that if the matrix (m) is composed of the numbers

$$x_{n+1}^{(n+2)} = -1; \qquad x_k^{(n+2)} = \cos \frac{2k-1}{2n}\pi, \quad k = 1, 2, \dots, n;$$

$$x_0^{(n+2)} = 1, \quad n = 1, 2, \dots, \tag{2}$$

then the interpolation process $\{H_n(f)\}_{n=1}^{\infty}$, constructed for $f(x) = |x|$, diverges at the point $x = 0$. Since the matrix (2) is obtained from the matrix of Chebyshev nodes by adding the points $x = \pm 1$ as nodes, in connection with Fejér' s theorem this result seems unexpected. Quite naturally there arose the question whether there exists such a continuous function on $[-1, 1]$ for which the process $\{H_n(f)\}_{n=1}^{\infty}$, constructed at the nodes (2), diverges at all points of the interval $(-1, 1)$. The answer to this question is given by Theorem 1.

Theorem 1. *The interpolation process* $\{H_n(f)\}_{n=1}^{\infty}$*, constructed for* $f(x) = 1 - x^2$ *at the nodes (2), diverges at every point of the interval* $(-1, 1)$*.*

We outline the proof. For the nodes (2),

$$\begin{aligned} H_{n+2}(f, x) = {} & \frac{f(1)}{2}\,[1 - (2n^2 + 1)(x - 1)](x + 1)T_n'(x) \\ & - \frac{f(-1)}{2}\,[1 + (2n^2 + 1)(x + 1)](x - 1)T_n'(x) \\ & + \sum_{j=1}^{n} f(x_j)\left(1 + \frac{3x_j(x - x_j)}{1 - x_j^2}\right)\frac{4x_j(x^2 - 1)^2 T_n^2(x)}{(x^2 - x_j^2)(x_j^2 - 1)[T_n'(x_j)]^2}. \end{aligned}$$

Therefore, for $f(x) = 1 - x^2,\ n = 4p$,

$$H_{n+2}(f, x) = \frac{2(x^2 - 1)T_n^2(x)}{n^2}\sum_{j=1}^{2p}\frac{4x_j^4 - (2x^2 + 1)x_j^2 - x^2}{(1 - x_j^2)(x^2 - x_j^2)^2}.$$

Decomposing the fraction under the $\sum$ into partial fractions and taking into account the symmetry of the nodes, we obtain

$$H_{n+2}(f, x) = \frac{(1 - x^2)T_n^2(x)}{n^2}\left(\frac{3}{x^2 - 1}\sum_{j=1}^{n}\frac{1}{1 - x_j} + \sum_{j=1}^{n}\frac{1}{(x - x_j)^2} + \frac{3x}{1 - x^2}\sum_{j=1}^{n}\frac{1}{x - x_j}\right).$$

We now use the identities:

$$\sum_{k=1}^{n}\frac{1}{\sin^2\theta_k/2} = 2n^2, \qquad \theta_k = \frac{(2k - 1)\pi}{2n}, \tag{3}$$

$$\frac{T_n'(x)}{T_n(x)} = \sum_{j=1}^{n}\frac{1}{x - x_j}, \qquad \frac{T_n^2(x)(1 - x^2)}{n^2}\sum_{j=1}^{n}\frac{1}{(x - x_j)^2} = 1 - \frac{\sin 2n\theta\cos\theta}{2n\sin\theta}, \qquad x = \cos\theta.$$

Consequently,

$$H_{n+2}(f,x) = 1 - 3\cos^2 n\theta + \frac{\sin 2n\theta\cos\theta}{n\sin\theta}, \qquad x = \cos\theta. \tag{4}$$

Since the process is symmetric, we may assume that $0 \le x < 1$. Suppose that at $\bar{x} \in [0,1)$ the process (4) converges. Then

$$\lim_{n\to\infty} \sin^2 n\theta = 1 - \frac{\cos^2\theta}{3}, \qquad \cos\theta = \bar{x},$$

and this contradicts the following lemma.

Lemma. For any θ from $[0,\pi/2]$ one can find a sequence of natural numbers $n_1 < n_2 < \cdots$, $n_k \to \infty$, $k \to \infty$, such that the equality

$$\lim_{k\to\infty} \sin^2 n_k\theta = 0$$

holds.

Theorem 2. The interpolation process $\{H_n(f)\}_{n=1}^{\infty}$, constructed for $f(x) = x^2$ at the nodes (2), diverges at every point of the interval $(-1,1)$.

The proof follows from Theorem 1 and the equality

$$H_{n+2}(1-z^2,x) = 1 - H_{n+2}(z^2,x).$$

In connection with Theorem 2 it is of interest to investigate the convergence of the process $\{H_n(f)\}_{n=1}^{\infty}$, constructed at the nodes (2), for the function $f(x) = x$.

Theorem 3. The process $\{H_n(f)\}_{n=1}^{\infty}$, constructed for $f(x) = x$ at the nodes (2), diverges at all points of $(-1,1)$ except $x = 0$.

We outline the proof. For $f(x) = x$ the identity holds

$$R_n(x) \equiv x - H_{n+2}(f,x) = \sum_{k=0}^{n+1} (x - x_k) L_k^2(x),$$

where $\{L_k(x)\}$ are the fundamental Lagrange polynomials of the nodes (2). After elementary transformations we obtain that

$$R_n(x) = \frac{\sin 2n\theta\sin\theta}{2n} - \frac{3\cos\theta\sin^2\theta}{2}\cos^2 n\theta. \tag{5}$$

Hence, with the aid of the lemma formulated earlier, it follows that the process diverges at every point $x \ne 0$ of $(-1,1)$. The convergence of the process for $x = 0$ is obvious, since for $x = 0$ the right-hand side of (5) is equal to zero.

3°. The results of item 2° show that extending the Chebyshev nodes by adding the points $x = \pm 1$ as nodes substantially worsens the behavior of the Hermite-Fejér interpolation process. The question arises: does this situation occur for every system of nodes? From the following theorems it is clear that this is not always so.

Theorem 4. *The interpolation process* $\{H_n(f)\}_{n=1}^{\infty}$, *constructed for* $f(x) = x^2$ *at the nodes*

$$x_k^{(n+2)} = \cos k\pi/(n+1), \qquad k = 0, 1, \dots, (n+1); \quad n = 1, 2, \dots, \tag{6}$$

converges uniformly on $[-1, 1]$. *Moreover,*

$$|H_{n+2}(f, x) - x^2| \le 4/(n+1). \tag{7}$$

We outline the proof. For the nodes (6),

$$\begin{aligned} H_{n+2}(f, x) = \frac{U_n^2(x)}{(n+1)^2} \Bigg\{ & \frac{f(1)}{4} \left[1 - \left(1 + \frac{2}{3} n(n+2)\right)(x-1)\right](x+1)^2 \\ & + \frac{f(-1)}{4} \left[1 + \left(1 + \frac{2}{3} n(n+2)\right)(x+1)\right](x-1)^2 \\ & + \sum_{j=1}^{n} f(x_j) \left(1 - \frac{x_j}{x_j^2 - 1}(x - x_j)\right) \left(\frac{x^2 - 1}{x - x_j}\right)^2 \Bigg\}, \end{aligned}$$

$$U_n(x) = \sin(n+1)\theta / \sin\theta, \qquad x = \cos\theta.$$

It is obvious that it suffices to prove the theorem for $f(x) = 1 - x^2$. In this case

$$H_{n+2}(f, x) = \frac{(1 - x^2)^2 U_n^2(x)}{(n+1)^2} \sum_{j=1}^{n} \frac{1 + x x_j - 2x_j^2}{(x - x_j)^2}. \tag{8}$$

Hence, after elementary transformations using the differential equation for the Chebyshev polynomials of the second kind, we obtain

$$H_{n+2}(f, x) = \frac{1}{(n+1)^2} \left[\sin^2 n\theta - 2n \sin n\theta \cos(n+1)\theta \sin\theta + n^2 \sin^2\theta\right],$$

$$x = \cos\theta. \tag{9}$$

Estimate (7) follows immediately from (9).

Remark. From (8) and (9) follows the identity

$$\sin^2(n+1)\theta \sum_{j=1}^{n} \frac{1+\cos\theta_j \cos\theta - 2\cos^2\theta_j}{(\cos\theta - \cos\theta_j)^2} =$$

$$= n^2 - 2n\frac{\sin n\theta}{\sin\theta}\cos(n+1)\theta + \left(\frac{\sin n\theta}{\sin\theta}\right)^2, \qquad \theta_j = \cos\frac{j\pi}{n+1}.$$

Putting here $\theta = \pi/2$ and $n = 2m$, we obtain

$$\sum_{j=1}^{2m} \frac{1}{\cos^2 j\pi/(2m+1)} = 4m(m+1).$$

This identity is an analogue of the well-known identity (3) of M. Riesz (^{2}).

Theorem 5. *The interpolation process* $\{H_n(f)\}_{n=1}^{\infty}$, *constructed for* $f(x) \equiv x$ *at the nodes (6), converges uniformly on* $[-1,1]$. *Moreover,*

$$|H_{n+2}(f,x) - x| \leq 1/n.$$

The proof follows from the identity

$$x - H_{n+2}(f,x) = \frac{1}{2(n+1)^2}(\cos\theta\sin^2(n+1)\theta - (n+1)\sin 2(n+1)\theta\sin\theta).$$

This theorem should be compared with the result of G. Szegő ([4], p. 349), according to which the Hermite-Fejér process constructed for $f(x) \equiv x$ at the nodes

$$\left\{\cos\frac{k\pi}{n+1}\right\}_{k=1}^{n}, \qquad n = 1, 2, \dots,$$

diverges at the point $x = 1$.

Theorem 6. *The process* $\{H_n(f)\}_{n=1}^{\infty}$, *constructed for* $f(x) = |x|$ *at the nodes (6), converges at the point* $x = 0$. *Moreover,*

$$|H_{n+2}(f,0)| \leq C\ln n/n.$$

This theorem should be compared with the result in ([1]), according to which the Hermite-Fejér process constructed for $f(x) = |x|$ at the nodes (2) diverges at the point $x = 0$.

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Note: Figure translations are in progress. See original paper for figures.