LAGRANGIAN DESCRIPTION OF THE DYNAMICS OF TURBULENT MOTION

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Abstract

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LAGRANGIAN DESCRIPTION OF THE DYNAMICS OF TURBULENT MOTION

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In the statistical description of turbulence in an incompressible viscous fluid, Eulerian variables are most often used. For the study of a number of phenomena (turbulent diffusion, deformation of surfaces or lines consisting of fluid elements), a Lagrangian method of description proves necessary, making it possible to follow the motion of fluid particles—marked points of a volume, moving within this volume in accordance with the equations of hydromechanics.

1. The unknowns in the Lagrangian method are the coordinates $\mathbf{X}(\mathbf{x}, t)$ of the fluid particles at time t, which at $t = 0$ were at the points $\mathbf{x}$. The transition from derivatives with respect to Eulerian coordinates to the initial coordinates of the fluid particles $\mathbf{x}$ is carried out by means of the matrix

$$\frac{\partial x^i}{\partial X^k} = \frac{1}{2D} \frac{\partial X^{\alpha_1}}{\partial x^{\beta_1}} \frac{\partial X^{\alpha_2}}{\partial x^{\beta_2}} \varepsilon_{\alpha_1 \alpha_2 k} \varepsilon_{\beta_1 \beta_2 i}, \tag{1}$$

where $\varepsilon_{\alpha_1 \alpha_2 \alpha_3}$ is the completely antisymmetric tensor, and D is the determinant of the inverse matrix

$$D = \frac{1}{6} \frac{\partial X^{\alpha_1}}{\partial x^{\beta_1}} \frac{\partial X^{\alpha_2}}{\partial x^{\beta_2}} \frac{\partial X^{\alpha_3}}{\partial x^{\beta_3}} \varepsilon_{\alpha_1 \alpha_2 \alpha_3} \varepsilon_{\beta_1 \beta_2 \beta_3},$$

connected with the change of density $\rho(0)/\rho(t)$ in a particle and equal to unity in an incompressible fluid. The Navier–Stokes equations for an incompressible fluid, in which the pressure is expressed through the velocity,

$$\frac{dv^\alpha}{dt} = \frac{1}{4\pi} \int v^\beta(\mathbf{X}', t) v^\gamma(\mathbf{X}', t) T^{\alpha;\beta\gamma}(\mathbf{X} - \mathbf{X}')\, d\mathbf{X}' + \nu \Delta v^\alpha,$$

$$T^{\alpha;\gamma}(\mathbf{X} - \mathbf{X}') = \frac{\partial^3}{\partial X^\alpha \partial X^\beta \partial X^\gamma} \frac{1}{|\mathbf{X} - \mathbf{X}'|}, \tag{2}$$

in Lagrangian variables take the form

$$\ddot{X}^\alpha = \frac{1}{4\pi}\int \dot{X}^\beta(\mathbf{x}',t)\dot{X}^\gamma(\mathbf{x}',t)T^{\alpha;\beta\gamma}(\mathbf{X}(\mathbf{x}',t)-\mathbf{X}(\mathbf{x},t))\,d\mathbf{x}'+$$

$$+\frac{\nu}{4}\frac{\partial X^{k_1}}{\partial x^{i_1}}\frac{\partial X^{k_2}}{\partial x^{i_2}}\frac{\partial}{\partial x^{i_3}}\left(\frac{\partial X^{l_1}}{\partial x^{\alpha_1}}\frac{\partial X^{l_2}}{\partial x^{\alpha_2}}\frac{\partial \dot{X}^{\alpha}}{\partial x^{\alpha_3}}\right)\varepsilon_{\alpha_1\alpha_2\alpha_3}\varepsilon_{i_1i_2i_3}\varepsilon_{k_1k_2m}\varepsilon_{l_1l_2m}. \qquad (3)$$

The pressure in (2) is expressed through the velocity up to a harmonic function determined from the boundary conditions and not random. In (2) and (3) the corresponding terms have not been written out for simplicity.

The system (2) must be supplemented by the incompressibility equation

$$\frac{1}{6}\frac{\partial X^{\alpha_1}}{\partial x^{\beta_1}}\frac{\partial X^{\alpha_2}}{\partial x^{\beta_2}}\frac{\partial X^{\alpha_3}}{\partial x^{\beta_3}}\varepsilon_{\alpha_1\alpha_2\alpha_3}\varepsilon_{\beta_1\beta_2\beta_3}=1. \qquad (4)$$

2. Let us introduce the distribution functions of the coordinates and velocities $P_n(\mathbf{V}_1,\mathbf{X}_1,\mathbf{x}_1,\dots,\mathbf{V}_n,\mathbf{X}_n,\mathbf{x}_n;t)$, such that the probability that the velocities and coordinates of n fluid particles which at $t=0$ are at the points $\mathbf{x}_1,\dots,\mathbf{x}_n$, at time t are located in the intervals $d\mathbf{V}_1,d\mathbf{X}_1,\dots,d\mathbf{V}_n,d\mathbf{X}_n$, is

$$P_n\,d\mathbf{V}_1 d\mathbf{X}_1\dots d\mathbf{V}_n d\mathbf{X}_n.$$

The functions P_n are normalized as follows:

$$\int P_{n+1}\,d\mathbf{V}_{n+1}d\mathbf{X}_{n+1}=P_n, \qquad \int P_1\,d\mathbf{V}_1 d\mathbf{X}_1=1$$

and are continuous as $\mathbf{x}_i\to\mathbf{x}_k$, i.e.

$$\lim_{\mathbf{x}i\to\mathbf{x}_k}P_{n+1}=P_n\delta(\mathbf{V}_i-\mathbf{V}_k)\delta(\mathbf{X}_i-\mathbf{X}_k).$$

The derivation of the evolution equations and of the remaining additional conditions for P_n can be carried out by averaging, over an ensemble, the corresponding equations in the case of laminar flow

$$P_n=\prod_{i=1}^{n}\delta(\mathbf{X}_i-\mathbf{X}(\mathbf{x}_i,t))\,\delta(\mathbf{V}_i-\dot{\mathbf{X}}(\mathbf{x}_i,t)). \qquad (5)$$

Differentiating (5) with respect to time and using the equations of motion (3), one can obtain the system of coupling equations

$$\frac{\partial P_n}{\partial t}+\sum_{i=1}^{n}\frac{\partial P_n}{\partial X_i^\alpha}V_i^\alpha+\frac{1}{4\pi}\frac{\partial}{\partial V_i^\alpha}\int P_{n+1}V_{n+1}^\gamma V_{n+1}^3 T^{\alpha\beta\gamma}(\mathbf{X}_i-\mathbf{X}_{n+1})\times$$
$$\times\, d\mathbf{V}_{n+1}d\mathbf{X}_{n+1}d\mathbf{x}_{n+1}+\frac{\nu}{4}\frac{\partial}{\partial V_i^\alpha}\int\frac{\partial^6 P_{n+5}}{\partial x_{n+1}^{\beta_1}\partial x_{n+2}^{\beta_2}\partial x_{n+3}^{\alpha_3}\partial x_{n+4}^{\alpha_2}\partial x_{n+5}^{\beta_3}\partial x_{n+6}^{\alpha_1}}\times$$
$$\times\, X_{n+1}^{l_1}X_{n+2}^{l_2}X_{n+3}^{k_1}(X_{n+4}^{k_2}V_{n+5}^\alpha+2X_{n+5}^{k_2}V_{n+4}^\alpha)\times$$
$$\times\,\varepsilon_{\beta_1\beta_2\beta_3}\varepsilon_{\alpha_1\alpha_2\alpha_3}\varepsilon_{l_1l_2m}\varepsilon_{k_1k_2m}\prod_{k=n+1}^{n+5}\delta(\mathbf{x}_i-\mathbf{x}_k)\times$$
$$\times\, d\mathbf{X}_{n+1}d\mathbf{V}_{n+1}d\mathbf{x}_{n+1}\ldots d\mathbf{V}_{n+5}d\mathbf{X}_{n+5}d\mathbf{x}_{n+5}=0.$$

To the latter it is necessary to add the incompressibility condition, obtained with the aid of (4):

$$P_n=\frac{1}{6}\int\frac{\partial^3 P_{n+3}}{\partial x_{n+1}^{\alpha_1}\partial x_{n+2}^{\alpha_2}\partial x_{n+3}^{\alpha_3}}X_{n+1}^{\beta_1}X_{n+2}^{\beta_2}X_{n+3}^{\beta_3}\times$$
$$\times\,\varepsilon_{\alpha_1\alpha_2\alpha_3}\varepsilon_{\beta_1\beta_2\beta_3}\prod_{k=n+1}^{n+3}\delta(\mathbf{x}_i-\mathbf{x}_k)\,d\mathbf{X}_{n+1}d\mathbf{V}_{n+1}d\mathbf{x}_{n+1}\ldots d\mathbf{X}_{n+3}d\mathbf{V}_{n+3}d\mathbf{x}_{n+3}.$$

Furthermore, from the averaging of (5), by differentiating with respect to $\mathbf{x}_i$, one can obtain the so-called consistency condition, associated with the normalization of the distribution functions and reflecting the circumstance that

$$\int P_n d\mathbf{V}_i d\mathbf{X}_i$$

does not depend on the initial coordinate $\mathbf{x}_i$ of the particle,

$$\frac{\partial P_n}{\partial x_i^\beta}=-\frac{\partial}{\partial V_i^\alpha}\int\frac{\partial P_{n+1}}{\partial x_{n+1}^\beta}V_{n+1}^\alpha d\mathbf{V}_{n+1}d\mathbf{X}_{n+1}\delta(\mathbf{x}_i-\mathbf{x}_{n+1})d\mathbf{x}_{n+1}-$$
$$-\frac{\partial}{\partial X_i^\alpha}\int\frac{\partial P_{n+1}}{\partial x_{n+1}^\beta}X_{n+1}^\alpha d\mathbf{V}_{n+1}d\mathbf{X}_{n+1}\delta(\mathbf{x}_i-\mathbf{x}_{n+1})d\mathbf{x}_{n+1}.$$

Let us note that the P_n are functions symmetric with respect to permutations of the groups of arguments $\mathbf{X}_i, \mathbf{V}_i, \mathbf{x}_i$ and $\mathbf{X}_k, \mathbf{V}_k, \mathbf{x}_k$.

In conclusion, we give a simple relation between the Lagrangian distribution functions under consideration and the probability density $F_n(\mathbf{V}_1, \mathbf{X}_1, \ldots, \mathbf{V}_n, \mathbf{X}_n; t)$ of the fact that the velocities at fixed points

$\mathbf{X}_1, \dots, \mathbf{X}_n$ at time t lie in the intervals $d\mathbf{V}_1, \dots, d\mathbf{V}_n$ (the Eulerian method of describing turbulence).

This relation can be obtained by averaging an exact equality that holds for an arbitrary function φ:

$$\int \varphi[\mathbf{V}(\mathbf{X}_1, t), \mathbf{X}_1, \dots, \mathbf{V}(\mathbf{X}_n, t), \mathbf{X}_n] \, d\mathbf{X}_1 \dots d\mathbf{X}_n =$$

$$= \int \varphi[\dot{\mathbf{X}}(\mathbf{x}_1, t), \mathbf{X}(\mathbf{x}, t), \dots, \dot{\mathbf{X}}(\mathbf{x}_n, t), \mathbf{X}(\mathbf{x}_n, t)] \, d\mathbf{x}_1 \dots d\mathbf{x}_n,$$

where $\mathbf{V}(\mathbf{X}, t)$ is the Eulerian velocity field, which gives

$$\int \varphi[\mathbf{V}_1, \mathbf{X}_1, \dots, \mathbf{V}_n, \mathbf{X}_n] \, F_n \, d\mathbf{V}_1 \, d\mathbf{X}_1 \dots d\mathbf{V}_n \, d\mathbf{X}_n =$$

$$= \int \varphi[\mathbf{V}_1, \mathbf{X}_1, \dots, \mathbf{V}_n, \mathbf{X}_n] \, P_n \, d\mathbf{V}_1 \, d\mathbf{X}_1 \, d\mathbf{x}_1 \dots d\mathbf{V}_n \, d\mathbf{X}_n \, d\mathbf{x}_n;$$

since the function φ is arbitrary, P_n and F_n are related as follows:

$$F_n = \int P_n \, d\mathbf{x}_1 \dots d\mathbf{x}_n.$$

The integration over coordinates in the last formulas is carried out over the volume occupied by the fluid.

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Note: Figure translations are in progress. See original paper for figures.