THE NEMYTSKII OPERATOR IN SPACES GENERATED BY GENERALIZED FUNCTIONS

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Abstract

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MATHEMATICS

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THE NEMYTSKII OPERATOR IN SPACES GENERATED BY GENERALIZED FUNCTIONS

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1°. Let X be a space with a σ-finite complete nonatomic measure μ, with $0 \leq \mu X \leq \infty$; let S be the set of all measurable functions on X with values in $\bar{R} = [-\infty, \infty]$.

Fix in S two functions a and b, with $a(x) < b(x)$ for every x. Let to each x there correspond an interval Δ_x with endpoints $a(x)$ and $b(x)$, which may or may not be contained in Δ_x. We shall assume that the sets $\{x : a(x) \in \Delta_x\}$ and $\{x : b(x) \in \Delta_x\}$ are measurable. It follows from this that the set

$$X_u = \{x : u \in \Delta_x\}$$

is measurable for any $u \in \bar{R}$.

Let the function $g(u, x)$, with values in $\bar{R}$, be defined for $x \in X,\ u \in \Delta_x$, be continuous in u for almost every x, and measurable in x on X_u for every $u \in \bar{R}$. These conditions (Carathéodory conditions) ensure the measurability of the superposition $g(\varphi(\cdot), \cdot)$, if $\varphi \in S$ and $\varphi(x) \in \Delta_x$ for almost every x ([10]).

For any $D \subset S$ put

$$D(\Delta) = \{\varphi \in D : \varphi(x) \in \Delta_x \text{ almost everywhere (a.e.)}\}.$$

It follows from the preceding that the Nemytskii operator h, generated by the function g, $h\varphi = g(\varphi(\cdot), \cdot)$, acts from $S(\Delta)$ into S.

The Nemytskii operator in various function spaces has been studied in many works (see, for example, ([1–6])). In the present note it is studied in a broad class of spaces including the spaces $\mathcal{L}^p$ with weight, $0 < p \leq \infty$, Orlicz spaces (including in the sense of Zăănen), Orlicz–Nakano spaces ([11]), Musielak–Orlicz F-spaces ([13,14]), and others. Conditions are studied for the action, boundedness, and continuity of the operator h in these spaces. It is interesting to note that certain properties of the spaces under consideration are obtained here as simple consequences of theorems on the Nemytskii operator. The spaces in question are generated by the so-called pre-generalized functions and generalized functions. Their definitions follow below.

A function $M(u,x)$, $0 \le u \le \infty,\ x \in X$, with values in $[0,\infty]$, is called a pre-generalized function if for almost every x it is nondecreasing and left-continuous in u (in the topology of $\bar{R}$), and for each u the function $M(u,\cdot) \in S$, with $M(0,\cdot) \in \mathcal{L}(X)$, i.e. summable on X. Put

$$I_M\varphi = \int_X M[|\varphi(x)|, x]\,d\mu, \qquad P_M = \{\varphi \in S_f : I_M\varphi < \infty\},$$

$$P_M^\alpha = \{\varphi \in S_f : \alpha\varphi \in P_M\}, \qquad L^M = \bigcup_{\alpha>0} P_M^\alpha, \qquad L_f^M = \bigcap_{\alpha>0} P_M^\alpha,$$

where S_f is the set of a.e. finite functions from S, with functions coinciding a.e. being regarded as equal. It is not difficult to show that L^M is a vector space, and L_f^M is its subspace.

A pre-generalized function M is called a generalized function if $M(0,x) = 0$ and $M(\infty,x) > 0$ a.e. on X, while $M(+0,x) = 0$ a.e. on $\{x : d_M(x) > 0\}$, where

$$d_M(x) = \sup\{u \in [0,\infty] : M(u,x) < \infty\}.$$

If M is a generalized function, then L^M may $\left({}^{9,13,14}\right)$ be regarded as an F-space with F-norm $\|\varphi\|_M =$

$$= \inf\{\varepsilon > 0 :\ I_M(\varepsilon^{-1}\varphi) \leqslant \varepsilon\}.$$

Moreover, $\|\varphi_n\|_M \to 0$ if and only if $I_M(\alpha\varphi_n) \to 0$ for every $\alpha > 0$.

2°. Let M and Φ be pregenfunctions; A and B nonempty sets of positive numbers;

$$M_A = \bigcap_{\alpha\in A} P_M^\alpha,$$

$$\Phi^B = \bigcup_{\beta\in B} P_\Phi^\beta, \qquad \delta_x = \Delta_x \cap \{u :\ |u| \sup A \leqslant d_M(x)\}.$$

We shall assume that $M_A(\Delta)$ is nonempty. It is obvious that if $\varphi \in M_A(\Delta)$, then $\varphi(x) \in \delta_x$ a.e. Consequently, δ_x is nonempty for almost every x. In analogy with the set $D(\Delta)$, define $D(\delta)$. From the preceding it follows that $M_A(\Delta) = M_A(\delta)$.

Theorem 1 (main). *The operator h acts from $M_A(\Delta)$ into Φ^B, i.e. $h[M_A(\Delta)] \subset \Phi^B$, if and only if there exist $\alpha \in A$, $\beta \in B$, $\gamma \geqslant 0$, and $f \in \mathcal{L}(X)$ such that, for all $x \in X$ and $u \in \delta_x$,*

$$\Phi[\beta|g(u,x)|, x] \leqslant \gamma M[\alpha|u|, x] - f(x). \tag{1}$$

Corollary 1. *If $h[M_A(\Delta)] \subset \Phi^B$, then $h[P_M^\alpha(\delta)] \subset P_\Phi^\beta$ for some $\alpha \in A,\ \beta \in B$.*

This assertion can be strengthened. To this end put

$$W(\varphi_0, \alpha, \rho) = \{\varphi :\ I_M(\alpha\varphi) \leqslant I_M(\alpha\varphi_0) + \rho\}.$$

Theorem 1′. *If $h[M_A(\Delta) \cap W(\varphi_0, \alpha_0, \rho_0)] \subset \Phi^B$, where $\varphi_0 \in M_A(\Delta)$, $\alpha_0 \in A$, $\rho_0 > 0$, then $h[P_M^\alpha(\delta)] \subset P_\Phi^\beta$ for some $\alpha \in A,\ \beta \in B$.*

From Theorems 1 and 1′, by varying the sets A and B, one can obtain various criteria for the action of the operator h. We list some of them.

1) $h[P_M^\alpha(\Delta)] \subset L^\Phi$ if and only if, for some $\beta > 0$, $\gamma \geqslant 0$, and $f \in \mathcal{L}(X)$, (1) holds for all $x \in X$ and $u \in \Delta_x$. If

$$h[P_M^\alpha(\Delta) \cap W(\varphi_0, \alpha, \rho_0)] \subset L^\Phi,$$

where $\varphi_0 \in P_M^\alpha(\Delta)$ and $\rho_0 > 0$, then

$$h[P_M^\alpha(\Delta)] \subset P_\Phi^\beta$$

for some $\beta > 0$.

Put

$$X^0 = \{x :\ 0 < d_M(x) < \infty\}.$$

2) $h[L_M^f(\Delta)] \subset L^\Phi$ if and only if, for some $\alpha > 0$, $\beta > 0$, $\gamma \geqslant 0$, and $f \in \mathcal{L}(X \setminus X^0)$, (1) holds for all $x \in X \setminus X^0$ and $u \in \Delta_x$, and

$$\Phi[\beta|g(0,\cdot)|,\cdot] \in \mathcal{L}(X^0).$$

If $\mu X^0 = 0$ and

$$h[L_M^f(\Delta) \cap W(\varphi_0, \alpha_0, \rho_0)] \subset L^\Phi,$$

where $\varphi_0 \in L_M^f(\Delta)$, $\alpha_0 > 0$, and $\rho_0 > 0$, then

$$h[P_M^\alpha(\Delta)] \subset L^\Phi$$

for some $\alpha > 0$.

If $g(u,x) = u$ for $x \in X,\ u \in \Delta_x$, then from Theorems 1 and 1′ various embedding criteria follow.

3) $P_M(\Delta) \subset P_\Phi$ if and only if there exist $\gamma \geqslant 0$ and $f \in \mathcal{L}(X)$ such that

$$\Phi(|u|, x) \leqslant \gamma M(|u|, x) + f(x)$$

for all $x \in X$ and $u \in \Delta_x$.

4) If $\mu X^0 = 0$ and

$$\left[L_M^f \cap \{\varphi :\ I_M(\alpha_0\varphi) \leqslant \rho_0\}\right] \subset L^\Phi,$$

where $\alpha_0 > 0$ and $\rho_0 > 0$, then $L^M \subset L^\Phi$.

5) $L^M \subset L^\Phi$ if and only if $P_M \subset L^\Phi$, i.e. if and only if, for some $\beta > 0$, $\gamma \geqslant 0$, and $f \in \mathcal{L}(X)$,
$$\Phi(\beta u, x) \leqslant \gamma M(u, x) + f(x)$$
for all $u \geqslant 0$ and $x \in X$. This is a generalization of Theorem 1 from ([11]) (see also ([12]), Lemma 3).

6) $L^M = P_M$ if and only if $P_M \subset P_M^2$, i.e. if and only if there exist $\gamma \geqslant 0$ and $f \in \mathcal{L}(X)$ such that
$$M(2u, x) \leqslant \gamma M(u, x) + f(x)$$
for all $u \geqslant 0$ and $x \in X$ (cf. ([4,12])).

3°. Let M be a genfunction, and let L^M be an F-space with F-norm $\|\cdot\|_M$. With the aid of Theorem 1 it is easy to determine the topological nature of the set P_M (cf. ([7,8])).

Indeed, as is not difficult to show, $\operatorname{int} P_M = M^{(1,\infty)}$. Consequently, P_M is open if and only if $P_M \subset P_M^\alpha$ for some $\alpha > 1$, and this is equivalent to the coincidence of L^M with P_M.

Similarly,
$$\overline{P}_M = M^{(0,1)}.$$
Consequently, P_M is closed if and only if there exist $\alpha \in (0,1)$, $\gamma \geqslant 0$, and $f \in \mathcal{L}(X)$ such that
$$M(u, x) \leqslant \gamma M(\alpha u, x) + f(x)$$
for all $x \in X$ and $u \in [0, d_M(x)]$. Another criterion is also valid: P_M is closed if and only if $P_M =$

$$= \{\varphi \in L^M : \ |\varphi(x)| \leq d_M(x) \text{ a.e.}\}.$$
Consequently, if $\mu X^0 = 0$, then P_M is closed if and only if $L^M = P_M$.

The theorems on boundedness of the Nemytskii operator proved in ([6]) for Orlicz spaces remain valid. Namely, let M and Φ be generating functions. Then, if $h(P_M) \subset L^\Phi$, then
$$\sup\{\|h\varphi\|_\Phi : \ \|\varphi - \varphi_0\|_M \leq r\} < \infty,$$
where $\varphi_0 \in L_M^f$ and $0 < r < 1$ (if $\varphi_0 = \theta$, one may take $r = 1$). Hence it follows that if $h(L^M) \subset L^\Phi$, then
$$\sup\{\|h\varphi\|_\Phi : \ \|\varphi\|_M \leq r\} < \infty$$
for every $r > 0$.

At the same time, if $\mu X^0 = 0$ and $h(L_M^f) \subset L^\Phi$, but h does not act from L^M into L^Φ, then there exists a set $E \subset L_M^f$, bounded in the F-norm $\|\cdot\|_M$, such that
$$\sup\{\|h\varphi\|_\Phi : \ \varphi \in E\} = \infty.$$

4°. Here we shall study conditions for continuity of the Nemytskii operator in spaces generated by generating functions. Depending on the choice of convergence in the spaces L^M and L^Φ, different types of continuity are obtained. Below we introduce a definition of continuity which covers many special cases.

Let $\varphi_0 \in L^M(\Delta)$, $h\varphi_0 \in L^\Phi$, where M and Φ are generating functions; A and B are nonempty sets of positive numbers.

Definition. Suppose the following condition is satisfied: if

$$I_M[\alpha(\varphi_n - \varphi_0)] \to 0$$

for every $\alpha \in A$, where $\varphi_n \in L^M(\Delta)$, $n = 1, 2, ...$, then

$$I_\Phi[\beta(h\varphi_n - h\varphi_0)] \to 0$$

for some $\beta \in B$. Then we shall say that

$$h \in (A, B, \varphi_0).$$

Put

$$g_0(v, x) = g(\varphi_0(x) + v, x) - g(\varphi_0(x), x), \qquad h_0\psi = g_0(\psi(\cdot), \cdot),$$
$$c_M(x) = \max\{u : \ M(u, x) = 0\}, \qquad p_\beta(v, x) = \Phi[\beta|g_0(v, x)|, x].$$

Obviously, the function g_0 is defined for $x \in X$ and $v \in \Delta_x^0 = \Delta_x - \varphi_0(x)$ (algebraic difference).

Theorem 2. If $\sup A < \infty$ and $\sup A \notin A$, then $h \in (A, B, \varphi_0)$ if and only if there exist $\alpha \in A$, $\beta \in B$, such that

$$h_0[P_M^\alpha(\Delta^0)] \subset P_\Phi^\beta$$

and, a.e. on $\{x : \ d_M(x) > 0\}$, the condition is satisfied:

$$p_\beta(v, x) = 0 \quad \text{for} \quad |v| \le (\sup A)^{-1} c_M(x)$$

and

$$\lim p_\beta(v, x) = 0 \quad \text{as} \quad |v| \to (\sup A)^{-1} c_M(x).$$

Theorem 3. If $\sup A = \bar\alpha \in A$, then $h \in (A, B, \varphi_0)$ if and only if there exists $\beta \in B$ such that

$$h_0[P_M^{\bar\alpha}(\Delta^0)] \subset P_\Phi^\beta$$

and, a.e.,

$$\lim p_\beta(v, x) = 0 \quad \text{as} \quad M(\bar\alpha|v|, x) \to 0.$$

In ([13,14]) the concept of ρ-convergence in modular spaces was studied.

In the space L^M, ρ-convergence of the sequence φ_n to φ_0 ($\varphi_n \xrightarrow{\rho} \varphi_0$) means that

$$I_M[\alpha(\varphi_n - \varphi_0)] \to 0$$

for some $\alpha > 0$. Consequently, ρ-continuity of the operator h at the point φ_0 (i.e., if $\varphi_n \xrightarrow{\rho} \varphi_0$, then $h\varphi_n \xrightarrow{\rho} h\varphi_0$) means that

$$h \in (\{\alpha\}, (0, \infty), \varphi_0)$$

for every $\alpha > 0$. Hence, from Theorem 3, it follows:

Corollary 2. The operator h is ρ-continuous at the point φ_0 if and only if

$$h[L^M(\Delta)] \subset L^\Phi$$

and, for every $\alpha > 0$, there exists $\beta > 0$ such that, a.e.,

$$\lim p_\beta(v, x) = 0 \quad \text{as} \quad M(\alpha|v|, x) \to 0.$$

In particular, if a.e. $c_M(x) = 0$ and $h[L^M(\Delta)] \subset L^\Phi$, then the operator h is ρ-continuous on all of $L^M(\Delta)$.

With the aid of Theorem 3, the question of the relation between convergence in the F-norm ($\|\varphi_n - \varphi_0\|_M \to 0$) and convergence in the mean ($I_M(\varphi_n - \varphi_0) \to 0$) is also easily resolved.

Corollary 3. Convergence in the F-norm in the space L^M coincides with convergence in the mean if and only if $L^M = P_M$ and, a.e.,

$$c_M(x) = 0$$

(cf. ([4,8])).

Theorem 4. If $\sup A = \infty$, then $h \in (A, B, \varphi_0)$ if and only if

$$h_0[P_M^\alpha(\Delta^0)] \subset P_\Phi^\beta$$

for some $\alpha \in A,\ \beta \in B$.

Corollary 4. The operator h is continuous (in the sense of convergence in the F-norm) at the point φ_0 if and only if, for every $\beta > 0$, there exists $\alpha > 0$ such that

$$h_0[P_M^\alpha(\Delta^0)] \subset P_\Phi^\beta.$$

All results from ([5]) concerning continuity of the Nemytskii operator carry over to spaces generated by generating functions. We give two new propositions.

Theorem 5. If $h[P_M(\Delta)] \subset L^\Phi$ and $\sup A = \infty$, then $h \in (A, B, \varphi)$ for every $\varphi \in P_M(\Delta)$ if and only if there exist $\alpha \in A$, $\beta \in B$, $\gamma \geq 0$, and $f \in \mathcal{L}(X)$ such that, for all $x \in X$, $u', u'' \in \Delta_x$,

$$\Phi[\beta|g(u'', x) - g(u', x)|, x] \leq \gamma\{M(|u'|, x) + M(\alpha|u'' + u'|, x)\} + f(x).$$

Theorem 6. Let $a \in L^M$ and $b \in L^M$ (see 1^0). If $h[L^M(\Delta)] \subset L^\Phi$ and h is continuous on $L^M(\Delta)$, then h is uniformly continuous on $L^M(\Delta)$.

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Note: Figure translations are in progress. See original paper for figures.