PROPERTIES OF SEQUENCES

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PROPERTIES OF SEQUENCES IN LOCALLY CONVEX SPACES

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In the present note we study properties of separable locally convex spaces (l.c.s.) which, in the case of Banach spaces, are usually called geometric. In the case of B-spaces, various assertions about bases and minimal systems play the instrumental role here; these carry over to l.c.s. in items 1, 3, 4. At the same time, the specificity of l.c.s. not only complicates the proof, but, in the main, leads to a "correct reading" of the theorems; this is especially clearly seen in the example of Theorem 6 (§ 3). At the same time, some results which have a purely geometric character in B-spaces acquire a topological meaning in l.c.s. (see the corollary to Theorem 8).

A sequence $\{x_k\}_1^\infty$ is called a **basis** if every element $x \in E = E(\{x_k\}_1^\infty)$ (where $E(\{x_k\}_1^\infty)$ denotes the closed linear span of the sequence $\{x_k\}_1^\infty$) admits a unique expansion in a convergent series:

$$x = \sum_{k=1}^{\infty} a_k x_k,$$

and **minimal** if there exists a sequence of continuous linear functionals $\{f_k\}_1^\infty \subset E^*$ (called conjugate) such that $f_k(x_i) = \delta_{ki}$. In those cases where it is necessary to use the uniform boundedness theorem, the class of barrelled spaces is employed (an l.c.s. in which every closed absolutely convex set that absorbs every point is a neighborhood of zero).

1. Criteria of basicity

Theorem 1. *Let E be a countably complete l.c.s. and let $\{x_k\}_1^\infty$ be a basis in E, and let the biorthogonal linear functionals $\{f_k\}_1^\infty$ ($f_k(x_j) = \delta_{kj}$) be bounded. Then the family of operators $\{U_n\}_1^\infty$, where*

$$U_n\left(\sum_1^\infty a_k x_k\right) = \sum_1^n a_k x_k,$$

is uniformly bounded.

The following theorem, for the case of Banach spaces, is the Banach basis criterion.

Theorem 2. *Let E be a complete barrelled space. A minimal complete sequence $\{x_k\}_1^\infty$ is a basis in E if and only if the family of operators*

$$\{U_n\}_1^\infty \left(U_n \left(\sum_1^\infty a_k x_k \right) = \sum_1^n a_k x_k \right)$$

is equicontinuous.

Necessity is obvious; the proof of sufficiency is based on the following lemma (for the case of normed spaces see ([1]), p. 63).

Lemma 1. *Let $\{U_n\}_1^\infty$ be an equicontinuous family of linear operators from the l.c.s. E_1 into the complete l.c.s. E_2. Then the set E of those elements x for which there exists $\lim_{n\to\infty} U_n x = U_0 x$ is linear and closed.*

By $\{\bar{n}_k\}_{k=1}^\infty$ we denote the sequence remaining from the natural sequence after deleting the sequence $\{n_k\}_{k=1}^\infty$.

Theorem 3*. Let $\{x_k\}_1^\infty$ lie in a complete barrelled space E. In order that the minimal sequence $\{x_k\}_1^\infty$ be an unconditional basis in $E = E(\{x_k\}_1^\infty)$, it is necessary and sufficient that for every sequence $\{n_k\}_{k=1}^\infty$

$$E(\{x_{n_k}\}_{k=1}^\infty) + E(\{x_{\bar{n}_k}\}_{k=1}^\infty) = E(\{x_n\}_1^\infty).$$

The use of the purely algebraic process of biorthogonalization, given in ([4]), p. 493, makes it possible to prove the following theorem, used in deriving the results of Section 2.

Theorem 4. Let E be a separable l.c.s. and let there exist a countable total set of continuous functionals $\{f_k\}_1^\infty \subset E^*$. Then in E there exists a complete minimal system with total conjugate system.

2. On the spaces $s(T)$ and $s^*(T)$. In many questions in the theory of l.c.s. a special role is played by the spaces $s(T)$ (the Tikhonov product T of one-dimensional spaces; for $T = \omega$ this means the product of a countable number of spaces) and its strong dual $s^*(T)$—the space of finite sequences with basis of cardinality T (if $\{e_\alpha\}_{\alpha\in T}$ is the natural basis in $s^*(T)$, which here may be the Hamel basis (see ([1]), p. 9), then $x \in s^*(T)$ means the existence of $\{a_k\}_{k=1}^n \subset T$ such that

$$x = \sum_{k=1}^n a_{a_k} e_{a_k}).$$

Proposition 1*.** a) Every linear subspace of $s^*(T)$ is closed.

b) If E_1 and E_2 are closed subspaces of a complete l.c.s. E, $E_1 \cap E_2 = 0$, and $E_1 \simeq s(T)$, then $E_1 + E_2$ is closed.

c) Every closed subspace of $s(T)$ is isomorphic to $s(T_1)$ for $T_1 \leq T$****, and, by virtue of b), the algebraic sum of any two closed subspaces is closed.

Let us note that the space $s^*(\omega)$ does not possess the property indicated in b), but, evidently, does possess the property indicated for $s(T)$ in c). The following theorem accounts for these facts.

Theorem 5. Let E_0 be an infinite-dimensional separable complete l.c.s. with a countable total system of functionals.

a) If for every l.c.s. E such that $E_0 \simeq E_1 \subset E$ and for every $E_2 \subset E$, $E_1 \cap E_2 = 0$, $E_1 + E_2$ is closed in E, then $E_0 \simeq s(\omega)$.

b) If in E_0, for any two subspaces E_1 and $E_2 \subset E_0$ ($E_1 \cap E_2 = 0$), $E_1 + E_2$ is closed (in this case we shall say that E_0 has no subspaces with zero inclination), then either $E_0 \simeq s(\omega)$, or $E_0^* \simeq s^*(\omega)$ (E^* is the dual in the strong topology); in particular, under the barrelledness condition on E_0, in the second case we obtain that $E_0^* \simeq s^*(\omega)$.

3. Selection of basic and minimal sequences. In the theory of B-spaces the following theorem is widely used ([6],[7]).

Let from a bounded sequence $\{x_k\}_1^\infty$ in a B-space it be impossible to select a subsequence strongly converging to zero, or weakly converging to $x_0 \neq 0$; then from $\{x_k\}_1^\infty$ one can select a basic subsequence.

In ([7]) this assertion is proved for metric l.c.s. However, as is easy—

* This theorem for the case of B-spaces is contained in ([2]). The absence of the concept of inclination in l.c.s. considerably complicates the proof.
** On different topologies in E^* see ([1]), p. 34.
*** Item a) is trivial; item b) see ([5]), p. 210.
**** What is meant is the inequality of cardinalities T_1 and T.

as is easy to see, the conditions of the assertion cannot be satisfied in nuclear (and complete) spaces, since in these spaces from any bounded sequence one can select a strongly convergent subsequence. The formulation below of an analogous result is free of this defect. Theorem 6 is used essentially in the proof of Theorem 8 of § 5.

Theorem 6. Let the sequence $\{x_n\}_1^\infty \subset E$ (a countably complete l.c.s.) be such that for any sequence of numbers $\{c_k > 0\}_{k=1}^\infty$ one cannot select from the sequence $\{c_{kx}k\}_{k=1}^\infty$ a subsequence weakly convergent to $x_0 \neq 0$. Then, for every seminorm p_α in E, one can select from the sequence $\{x_n\}_1^\infty$ a minimal subsequence $\{x_{n_k}\}_{k=1}^\infty$. If, moreover, E is of type (F) (metrizable) or (LF) (a countable inductive limit of metrizable spaces), then the subsequence $\{x_{n_k}\}_{k=1}^\infty$ can be chosen to be basic.

Remark 1. From a basic sequence in a countably complete l.c.s. one can select a minimal subsequence.

Remark 2. The conditions of Theorem 6 are necessary in the sense that they are fulfilled for every minimal subsequence with total conjugate on $E(\{x_k\}_1^\infty)$.

Remark 3. If the minimal system $\{x_k\}_1^\infty$ satisfies the conditions of Theorem 6, then the conjugate system $\{f_k\}_1^\infty$ is total on $E(\{x_k\}_1^\infty)$.

4. Stability of minimal and basic systems*

A sequence $\{x_n\}_1^\infty \subset E$ is called **stable** if for every bounded set $M \subset E$ there exists $\{\varepsilon_k > 0\}_{k=1}^\infty$ such that, for any $\{z_k\}_{k=1}^\infty \subset M$, the sequence $\{y_k = x_k + \varepsilon_k z_k\}_{k=1}^\infty$ is equivalent to $\{x_k\}_1^\infty$, i.e., the operator $A : Ax_k = y_k$ is an isomorphism of the spaces $E(\{x_k\}_1^\infty)$ and $E(\{y_k\}_1^\infty)$.

Theorem 7. Let E be a countably complete l.c.s. and let $\{x_k\}_1^\infty \subset E$ be an equicontinuously continuous (see (3)) basic (or minimal) sequence. Then either $E(\{x_k\}_1^\infty) \simeq s(\omega)$, or from $\{x_k\}_1^\infty$ one can select an infinite subsequence $\{x_{n_k}\}_1^\infty$ which will be a stable basis (respectively, a stable minimal system) in its envelope.

Theorem 7 is used in all results of § 5.

5. Some applications; zero inclination

A minimal system $\{x_\alpha\}_{\alpha\in A}$ is called a **generalized unconditional basis** of the space E if every element $x \in E$ admits a unique expansion into a series

$$x = \sum_\alpha a_\alpha x_\alpha,$$

convergent under any permutations, with $a_\alpha \neq 0$ for at most a countable subset of A.

The use of the stability criterion from § 4 makes it possible to prove the following.

Proposition 2. Let the l.c.s. and F-space E possess a generalized unconditional basis $\{x_\alpha\}_{\alpha\in A}$. Then every infinite-dimensional subspace $E_1 \subset E$ possesses an infinite-dimensional subspace $E_2 \subset E_1$ with an unconditional basis.

Theorem 8 (on zero inclination).** Let E be a countably complete l.c.s. and E_i $(i = 1, 2)$ closed subspaces. If $E_1 + E_2 \neq \overline{E_1 + E_2}$ (here closure is considered with ordinary, not generalized, sequences), then there exist infinite-dimensional closed subspaces $E_3 \subset E_1$ and $E_4 \subset E_2$ which are isomorphic: $E_3 \simeq E_4$.

* A criterion for the stability of basic sequences in F-spaces was studied in (8); a result close to the one given here is mentioned in (9); in another spirit, criteria for the stability of bases in l.c.s. are studied in (10).

** For the case of B-spaces this fact was discovered by V. I. Gurarii (11). The proof of the theorem in l.c.s. requires the use of the results of §§ 3 and 4.

Corollary. Let E_1 be an F-space and E_2 a countable inductive limit of B-spaces, $E_i \subset E$ $(i = 1, 2)$. If $E_1 + E_2 \neq E_1 + E_2$, then in E_1 and E_2 there exist infinite-dimensional subspaces isomorphic to a B-space.

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Note: Figure translations are in progress. See original paper for figures.