# AN EXTERNAL INVERSE PROBLEM FOR THE VOLUME POTENTIAL OF VARIABLE DENSITY FOR A BODY CLOSE TO A GIVEN ONE

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## Abstract

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*MATHEMATICS*

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# AN EXTERNAL INVERSE PROBLEM FOR THE VOLUME POTENTIAL OF VARIABLE DENSITY FOR A BODY CLOSE TO A GIVEN ONE

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**1°.** In this paper a solution is given of the inverse problem for the metaharmonic potential in the following formulation (([1,7]; see also ([1,2,6–8]), where the question of uniqueness and stability of the external problem was investigated).

One seeks a body $T_1$ such that its external metaharmonic potential ($\chi \geqslant 0$) of a given variable density $\mu$ is equal outside the body $T_1$ to a given metaharmonic function $H$ ($H$ is a regular solution of the equation $\Delta H - \chi^2 H = 0$), $H$ decreases at infinity as a metaharmonic potential and is close, in the sense of a certain functional metric, to the external metaharmonic potential $V$ of the given body $T$ of density $\mu$. It is assumed that the boundary $S$ of the body $T$ belongs to the class $A^{(2,\lambda)}$.

In the present article, for the metaharmonic potential ($\chi \geqslant 0$), the existence and uniqueness of the solution of the indicated problem are proved. For the Newtonian potential ($\chi = 0$), in the case of constant density $\mu = 1$, a problem of this kind for stellar bodies was solved by V. K. Ivanov ([1]), and under the assumption that the body $T$ is a sphere, a similar problem was studied by L. N. Sretenskii ([8]).

We note that the main result obtained in this paper is new also for the Newtonian potential ($\chi = 0$): first, a general class of variable densities is considered; second, there are no restrictions of the "stellar" type on the body $T$. This remark also applies to the metaharmonic potential ($\chi \geqslant 0$), for which a similar problem was studied in the case of a constant density of a stellar body $T$ in the author' s work ([7]).

**2°.** By virtue of the smoothness conditions on the surface $S$, for sufficiently small $v$ ($|v| \leqslant 3\varepsilon_0,\ \varepsilon_0 > 0$) every point $y = (y_1, y_2, y_3)$ of the three-dimensional space $E^3$, by the formula

$$\mathbf{y} = \mathbf{x} + v\mathbf{n}_x \tag{1}$$

can be defined in a unique way in a neighborhood of the surface $S$ with the aid of three curvilinear coordinates $(\xi, \eta, v)$, where $(\xi, \eta, 0)$ are the curvilinear coordinates of the point $\mathbf{x}$ of the surface $S$; $\mathbf{n}_x$ is the unit vector of the exterior normal to the surface $S$ at the point $\mathbf{x}$. Denote the metaharmonic potential $(\chi \geqslant 0)$ of a set $A$, $A \subset E^3$, with density $\mu(y) \neq 0$ almost everywhere for $y \in A$, by

$$V(x; A, \mu) = \int_A \mu(y) \frac{e^{-\chi r_{xy}}}{r_{xy}}\, dy,$$

where $r_{xy} = |y - x|$ is the distance between the points $y$ and $x$. Consider a bounded simply connected domain $T$, bounded by a surface $S$ of class $A^{(2,\lambda)}$. Let the external metaharmonic potential $V(x; T, \mu)$ of the body $T$ of density $\mu$ be known. In addition, suppose that outside a domain $T_0$, lying inside $T$ at a positive distance $d$ from the boundary $S$, a metaharmonic function $H$ is given which at infinity behaves like a metaharmonic potential.

In addition we shall assume that:

1) $\mu(y)$ is a given real analytic function in the domain $D'$ ($D' \supset T + S$, the boundary of $D'$ is at a positive distance, greater than $3\varepsilon_0$, from the surface $S$) and $\mu(y)$ is nowhere equal to zero on the surface $S$.

2) Each of the quantities

$$\left\| \frac{\partial H}{\partial \nu} - \frac{\partial V(T, \mu)}{\partial \nu} \right\|, \qquad \left\| \frac{\partial^2 H}{\partial \nu^2} - \frac{\partial^2 V(T, \mu)}{\partial \nu^2} \right\|$$

does not exceed $\omega C$, where $0 < \omega < d$, $C = C(T)$, $\omega = \omega(T, \mu, \varepsilon_0, d)$, and the norm $\| \cdot \|$ of the limiting, from outside $T$, values of

$$\frac{\partial}{\partial \nu}(H - V), \qquad \frac{\partial^2}{\partial \nu^2}(H - V)$$

is equivalent to the norm in the space $C^{(1,\lambda)}(s)$.

Let $\{S_1\}$ be the class of surfaces whose equation in the curvilinear system of coordinates (1) has the form

$$\{v = \zeta(\xi, \eta)\}, \qquad |v| \leqslant \varepsilon_0, \qquad \zeta \in C^{(1,\lambda)}.$$

Under these conditions, for the body $T$, the surfaces $S$ and $S_1$, and the functions $V$ and $H$, the following holds.

**Theorem.** *There exists, and moreover is unique, a surface $S_1$ bounding a body $T_1$, satisfying the condition $\|\xi\| < d$, such that the external metaharmonic* *potential $V(x;T_1,\mu)$ of the body $T_1$ of the given density $\mu$ is equal to the given metaharmonic function $H$ in the domain exterior to the surface $S_1$, i.e.*

$$H(x) = V(x;T_1,\mu) \quad \text{for } x \in E^3 \setminus \overline{T}_1.$$

$3^0$. The derivation of the nonlinear integro-differential equation determining the boundary of the sought body $T_1$ is based on the ideas of V. K. Ivanov ([1]), L. Lichtenstein ([3]), and A. M. Lyapunov ([4]). It should be noted that the arguments used in the theory of nonhomogeneous equilibrium figures ([3,4]), for our problem, lead to an integral equation of the first kind, which cannot be investigated even in the case $\mu = 1$, as is noted in ([1]). Therefore, following the method of V. K. Ivanov ([1]), we use the normal derivative of the potential, although for our case, owing to the variable density, the derivation of the principal nonlinear equation is carried out somewhat differently than in ([1]).

Denote by $D$ the domain bounded by the surface $S_{2\varepsilon_0}$, which in the curvilinear system of coordinates (1) is defined by the equation

$$v = 2\varepsilon_0.$$

The derivation of the equation and its investigation are based on the following

**Lemma.** *Let $T$ be a simply connected domain bounded by a surface $S \in A^{(2,\lambda)}$; let $\hat{T}$ be a simply connected domain bounded by a surface $\hat{S}$, defined in the curvilinear system of coordinates (1) by the equation*

$$v = \zeta(\xi,\eta), \qquad |v| \leqslant \varepsilon_0;$$

*$L$ be the domain with boundary $S_{2\varepsilon_0}$, $D$ contain $T+S$ and $\hat{T}+\hat{S}$. There exist positive numbers $\Omega$ and $K$, depending only on the shape of $T$, such that for $\|\xi\| \leqslant \Omega$ one can construct a continuous one-to-one mapping of the form*

$$\hat{y}_k = y_k + \alpha_k(y_1,y_2,y_3) \qquad (k=1,2,3)$$

*of the domain $D$ onto some domain $\hat{D}$, which maps $T+S$ onto $\hat{T}+\hat{S}$, and moreover the Jacobian $\hat{J}(y)$ of the mapping*

$$\hat{J}(y) = \partial(\hat{y}_1,\hat{y}_2,\hat{y}_3)/\partial(y_1,y_2,y_3) \neq 0 \qquad \text{for } y \in D;$$

*the functions $\alpha_k(y) \in C^{(1,\lambda)}(D)$ $(k=1,2,3)$ and satisfy the condition*

$$|\alpha_k|,\ |\partial\alpha_k/\partial y_i|,\ |\partial\alpha_k/\partial y_i|_\lambda \leqslant \Pi,$$

*where $\Pi$ is a positive number that may be taken equal to*

$$K\|\xi\| = \Pi,$$

and, moreover, for the functions $a_k$ the conditions are satisfied

$$a_k\big|_{\nu=0} = a_k(x) = \zeta(x)a_k(x), \qquad x \in S; \quad \mathbf{n}_x = \{a_1, a_2, a_3\}; \qquad \partial a_k/\partial\nu\big|_{\nu=0} = 0.$$

4°. Introduce the function

$$F_t^\varepsilon = \int_{T_t'} \mu(y_t) \frac{\partial}{\partial\nu}\left(\frac{e^{-\chi|y_t - z|}}{|y_t - z|}\right) dy_t\bigg|_{\nu=t\zeta+\varepsilon},$$

where $0 < \varepsilon \le \varepsilon_0$, the point $z$ has curvilinear coordinates $(\xi, \eta, \nu)$ ($(\xi, \eta, 0)$ are the curvilinear coordinates of the point $x \in S$); $T_t$ is the domain bounded by the surface $S_t$, depending on the parameter $t$, $S_t$ being defined by the equation

$$\nu = t\zeta(\xi, \eta), \qquad 0 \le t \le 1.$$

Expanding the function $F_t^\varepsilon$ in a series in powers of $t$, putting $t = 1$, and letting $\varepsilon$ tend to zero, we obtain for the function $\zeta(\xi, \eta)$, which determines the boundary of the sought body $T_1$, the nonlinear integro-differential equation

$$A(\zeta\mu) = \frac{\partial}{\partial\nu}V(T, \mu)\bigg|_{\nu=0} + \zeta\frac{\partial^2 V(T,\mu)}{\partial\nu^2}\bigg|_{\nu=0} - \frac{\partial H}{\partial\nu}\bigg|_{\nu=\zeta} + \Psi(\zeta), \tag{2}$$

where $\frac{\partial}{\partial\nu}V(T, \mu)$, $\frac{\partial^2 V(T,\mu)}{\partial\nu^2}$ are the exterior limiting values on the surface $S$ of the metaharmonic potential $V(T, \mu) = V(x; T, \mu)$ of the body $T$ with density $\mu$,

$$A(\zeta\mu) = 2\pi\zeta(x)\mu(x) - \int_S \frac{\partial}{\partial n_x}\frac{e^{-\chi r_{xy}}}{r_{xy}}\,\mu(y)\zeta(y)\,dS_y, \qquad x \in S,$$

$$\Psi(\zeta) = \sum_{n=2}^{\infty}\left[\lim_{\varepsilon\to 0}\frac{1}{n!}\frac{d^n}{dt^n}F_t^\varepsilon\bigg|_{t=0}\right].$$

We note that every sufficiently small solution of equation (2) is a solution of the problem.

For the operator $\Psi(\zeta)$ estimates hold that are similar to those proved in theorem 3 of the author's paper ($7a$) (see also ($1, 7b$)). In view of the indicated estimates, under the conditions of the main theorem the existence and uniqueness of a solution of equation (1) are proved, and this solution can be found by the method of successive approximations.

**Remark.** The posed problem has, moreover, a unique solution under the assumption that the body $T$ consists of a finite number of simply connected finite domains $T_1, \ldots, T_m$ with boundaries $S_1, \ldots, S_m$ of class $A^{(2,\lambda)}$ (with $T_k + S_k \neq T_j + S_j,\ j \neq k$); by $S$, the boundary of the body $T$, we denote the union of the boundaries $S_1, \ldots, S_m$. Instead of the domain $T_0$, one considers a body $T^0$ consisting of simply connected domains $T_1^0, \ldots, T_m^0$, lying respectively inside $T_1, \ldots, T_m$ at a positive distance $d$ from $S_1, \ldots, S_m$. The function $H$ is a regular solution of the metaharmonic equation in the domain $E^3 \setminus \overline{T}$ with the corresponding decrease at infinity. It is assumed that $\mu(y)$ is the collection of given functions $\mu_k$, analytic in the domains $D'_k = T_k + S_k$ ($k = 1, 2, \ldots, m$), each function $\mu_k(y)$ being nowhere zero on the surface $S_k$.

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*Note: Figure translations are in progress. See original paper for figures.*