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Abstract

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Yu. E. MURAKHVER

ON THE VELOCITY OF A WAVE FRONT

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1. The velocity of a wave front is of interest in various physical applications. A number of problems concerning its determination have been solved by analytically continuing the Fourier transform of the wave function into the complex plane (see, for example, ([1,2])). In the present paper, by this method, some results of a rather general form are obtained and concrete examples are considered. The article is connected with an earlier work of the author ([3]), where the spreading of wave packets was studied by entirely different means.

We shall call $z = a$ the left front of the function (wave) $G(z)$, if $G(z) = 0$ for $z < a$, but there exists an $\alpha > 0$ such that $G(z) \neq 0$ almost everywhere on the interval $[a, a + \alpha]$. The right front of a wave is defined analogously. We shall denote wave functions by capital letters, and their Fourier transforms by the corresponding small letters:

$$g(k) = \int_{-\infty}^{\infty} G(z)e^{ikz}\, dz.$$

All integrals are understood in the Lebesgue sense.

From the above definition of the wave front it is clear that we restrict ourselves to the study only of finite and semi-finite (there exists one front) functions. This does not exhaust all cases encountered in physics. However, even in more complicated cases one can often reduce the problem to the one considered by us, if, as $G(z)$, one takes not the complete wave, but a certain perturbation (which has been prepared during a finite time).

2. The principal mathematical fact used in the present paper is the following.

Theorem 1. *The following two assertions are equivalent:*

A. $z = a$ and $z = b$ are, respectively, the left and right fronts of the function $G(z) \in L_2[a, b]$.

B. $g(k)$ can be represented in the form

$$g(k) = e^{iak} g_+(k) = e^{ibk} g_-(k), \tag{1}$$

where $g_\pm(k)$ are entire functions of first order, which are Fourier transforms of finite functions from L_2 ([4]) and satisfy the conditions

$$\lim_{|k|\to\infty,\ \mathrm{Im}\,k>0} g_+(k) = 0; \qquad \overline{\lim_{|k|\to\infty,\ \mathrm{Im}\,k>0}} \left|e^{-i\eta k} g_+(k)\right| = \infty; \tag{2}$$

$$\lim_{|k|\to\infty,\ \mathrm{Im}\,k<0} g_-(k) = 0; \qquad \overline{\lim_{|k|\to\infty,\ \mathrm{Im}\,k<0}} \left|e^{i\eta k} g_-(k)\right| = \infty \tag{3}$$

(η is an arbitrary positive number; $\arg k$ is considered fixed).

Let us first prove that A implies B. Denoting $g_+(k) = g(k)e^{-iak}$ and estimating this function in modulus, we write

$$|g_+(k)| \le \int_0^{b-a} |G_1(z)| e^{-pz}\, dz,$$

where $G_1(z) = G(z+a)$ and $p = \mathrm{Im}\, k$. By a known property of the Lebesgue integral, for almost all z there exists the limit

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_z^{z+\varepsilon} |G_1(x)|\, dx = |G_1(z)|.$$

Let the limit exist also at $z = 0$ (if this is not so, the proof is changed slightly, but the result remains the same). Then, by the property ···

of the Laplace transform (see, for example, (5), p. 31),

$$\lim_{p\to\infty} \left\{ p \int_0^{b-a} |G_1(z)| e^{-pz}\, dz \right\} = |G_1(0)|.$$

Consequently, $g_\pm(k) \to 0$ as $|k| \to \infty$ and $\mathrm{Im}\, k > 0$.

Next consider the function

$$\tilde{g}_+(k) = \exp[-i(a+\eta)k] \int_a^{a+\eta} G(z) e^{-ikz}\, dz,$$

where $\eta < a$ (a occurs in the definition of the wave front). We shall prove that $\tilde{g}_+(k)$ is unbounded along any ray in the upper half-plane. Suppose the contrary. Let $\tilde{g}_+(k)$ be bounded along the ray $\arg k = \varphi_0$ ($0 < \varphi_0 < \pi$). Noting that this function is bounded in the lower half-plane, and applying the Phragmén–Lindelöf theorem (see, for example, (6), vol. 2, Ch. 7) to domains bounded by the ray $\arg k = \varphi_0$ and one of the two rays $\arg k = \varphi_0 + \pi \pm \varepsilon$ (here we use the fact that $\tilde{g}_+(k)$ is an entire function of order one), we arrive at the conclusion that $\tilde{g}_+(k)$ is bounded in the upper half-plane, i.e. $\tilde{g}_+(k) = \text{const} = 0$ by the Liouville and Riemann–Lebesgue theorems. But this contradicts the condition that $z = a$ is the wave front.

Since the function

$$\int_{a+\eta}^{b} G(z) e^{ikz} \, dz \cdot \exp[-i(a+\eta)k]$$

is bounded for $\operatorname{Im} k > 0$, formulas (2) have been proved. Formulas (3) are proved analogously.

To see further that B implies A, it suffices to use Theorem X from (4) and what was proved above.

The following theorem is not used directly in the present paper, but turns out to be useful in the problem of a boundary perturbation. For example, all the results of (2) are obtained from it immediately.

Theorem 2. *The following two assertions are equivalent:*

A. $z = a$ is the left front of the function $G(z) \in L_2[a, \infty)$.

B. $g(k) = e^{ika} g_+(k)$, where $g_+(k)$ is analytic in the upper half-plane and satisfies there the conditions (with $\arg k$ fixed)

$$\lim_{|k|\to\infty} g_+(k) = 0, \qquad \overline{\lim_{|k|\to\infty}} \left| e^{-i\eta k} g_+(k) \right| = \infty. \tag{4}$$

The theorem is proved analogously to the preceding one.

Theorem 1 gives, up to a pre-exponential factor, the character of the variation of $g(k)$ along any ray in the complex plane. Consequently, it generalizes the Wiener–Paley Theorem X (4), which gives only an upper bound for the growth of $g(k)$, and moreover not along an arbitrary ray, but only for the complex plane as a whole.

3. With the aid of Theorem 1 one can obtain a complete solution of the problem of the velocity of the fronts of the most commonly used wave packets:

$$F(z,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\{i[\omega(k)t - kz]\} f_0(k) \, dk, \tag{5}$$

$$F(z,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[e^{i\omega(k)t} f_1(k) + e^{-i\omega(k)t} f_2(k) \right] e^{-ikz} \, dk. \tag{6}$$

In formulas (5)–(6), $\omega(k)$ and $-\omega(k)$ are branches of the dispersion curve; no restrictions are imposed on $f_0(k)$, $f_1(k)$, $f_2(k)$, and $\omega(k)$, except those that follow from the finiteness of $F(z,t)$.

Theorem 3. *In order that both wave fronts (5) propagate with finite velocity for arbitrary finite $F_0(z) = F(z,0) \in L_2$, it is necessary and sufficient that $\omega(k)$ have the form $ck + \gamma$, where c and γ are constants, with c real (it is easy to verify that c coincides with the velocity of the wave front). In other words, the velocity of the fronts of the wave packet*

(5) will be finite if and only if dispersion is absent and the damping coefficient does not depend on k.

Theorem 4. In order that both wave fronts (6) propagate with finite velocity for arbitrary $F_0(z)$ and $G_0(z) = \partial F/\partial t|_{t=0}$ belonging to L_2 on a finite interval of the real axis and being finite functions, it is necessary and sufficient that the equality $\omega(k) = c\sqrt{k^2 + pk + q}$ hold, where c, p, and q are constants, with c real (c coincides with the velocity of the wave front).

The proof of these theorems is based on the explicit form of $f(k,t)$. For example, for the wave packet (6), $f(k,t)$ is an entire function of first order in the variables k and t; the same can be said of $\partial^2 f/\partial t^2$. But $\partial^2 f/\partial t^2 = -\omega(k)^2 f(k,t)$, whence it follows that $\omega(k)^2$ is an entire function (as can be shown, it is not meromorphic, in view of the arbitrariness of $F_0(z)$ and $G_0(z)$).

For what follows we write

$$f(k,t) = f_0(k)\cos\omega(k)t + ig_0(k)\sin\omega(k)t/\omega(k) \tag{7}$$

($f_0(k)$ and $g_0(k)$ are entire functions of first order), which is easily done by differentiating (6) with respect to t under the integral sign.

It is not difficult to prove that $\omega(k)^2$ is a polynomial. Indeed, suppose that $\omega(k)^2$ is a transcendental function. Then, by the Sokhotskii-Casorati theorem (see, for example, (6), vol. I, ch. 4), there exists a sequence k_n ($|k_n| \to \infty$) such that $\omega(k_n)^2 - k_n^4 \to 0$. But this means (in view of the arbitrariness of $F_0(z)$ and $G_0(z)$) that $f(k,t)$ cannot be an entire function of first order. We have arrived at a contradiction, which proves the assertion formulated. Proceeding in an analogous way, one can prove that $\omega(k)^2$ does not exceed some polynomial of the second degree, and hence itself is a polynomial of this degree. Consequently, $\omega(k) = c\sqrt{k^2 + pk + q}$. But, by virtue of Theorem 1, the constant c is real. Thus necessity is proved. Sufficiency is easily established by using (7) and Theorem 1.

4. As a simple example, let us consider the Klein-Fock-Gordon equation for a free particle. In this case there are two branches $\omega(k) = \pm c\sqrt{k^2 + k_0^2}$ (the same dispersion law holds for various types of waves in a plasma, for example for Langmuir waves), where $k_0 = mc/\hbar$. By Theorem 4 the velocity of the wave fronts is equal to c. At the same time, by Theorem

3, a wave packet composed of harmonics of one branch has an infinite velocity for at least one of the fronts, i.e., from harmonics of one branch one cannot construct a wave packet that is finite for $t \geq 0$. This is easy to verify directly from formula (7).

Let us now study the heat-conduction equation in more detail. We have $\omega(k) = iDk^2$, where D is the thermal diffusivity coefficient. By Theorem 3 the velocity of the front of a thermal wave is infinite, as is well known. It is easy to show that this fact is due to Fourier' s equation

$$\mathbf{q} = -\chi \nabla T \tag{8}$$

($\mathbf{q}$ is the heat-flux vector). Indeed, the fundamental solution of the heat-conduction equation has the form

$$G(z,t) = \exp[-z^2/4l^2](4\pi l^2)^{-1/2},$$

where $l = \sqrt{Dt}$. Hence, for example,

$$-\frac{\partial^2 G}{\partial z^2}\, l \Big/ \frac{\partial G}{\partial z} = \frac{z}{2l} - \frac{l}{z} \gg 1$$

for $z \gg l$. Thus, for $z > l$, higher derivatives with respect to the coordinates, not taken into account by equation (8), are essential. One might have tried to obtain a finite front velocity by including them in (8). In that case, for $\omega(k)$ we would obtain a polynomial whose degree is equal to the order of the highest derivative in Fourier' s equation, i.e., by Theorem 3 the velocity of the wave front would remain infinite.

Let us now formally replace (8) by the nonlocal equation

$$q(\mathbf{r}) = -\iiint G(\mathbf{r}-\mathbf{r}')T(\mathbf{r}')\, d\mathbf{r}'.$$

Proceeding as in the derivation of the usual heat-conduction equation, we obtain (for the one-dimensional case) $\omega(k) = -\frac{1}{c}kg(k)$, where c is the specific heat. By virtue of Theorem 3 the velocity of the wave front can be finite only when $g(k) = \text{const}$, which, as is not hard to see, is meaningless.

We see that replacing equation (8) by a nonlocal equation without retardation in t leaves the velocity of the wave front infinite, as was to be expected from simple physical considerations. Thus, in order to overcome this difficulty there are only two possibilities: the use of a nonlocal equation with retardation, or taking nonlinear effects into account.

5. Let us now summarize the main results of the present work.

1) Theorems 3 and 4 show that the requirement that the wave-front velocity be finite imposes very severe restrictions on the admissible dispersion laws, and that wave packets composed of separate branches have physical meaning only in the trivial case $\omega(k) = ck + \gamma$ ($\operatorname{Im} c = 0$).

2) If the number of branches is large, and especially if it is infinite, and also if $\omega(k) = ck + \gamma$ (see Theorem 3), the selection of separate branches may prove inexpedient or even impossible. In this case one may simply write

$$F(z,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(k,t) e^{ikz} \, dk,$$

and, in order to determine the front velocity, use Theorem 1 directly. This requires only knowledge of the asymptotics of $f(k,t)$ in the complex k-plane.

3) The generalization of Theorems 1 and 2 to the multidimensional case is not obvious; as for Theorems 3 and 4, they, as is easy to show, remain valid, with the role of k being played by the modulus of the wave vector.

It may seem that the example considered in the book [1] contradicts Theorem 3. There a finite wave-front velocity (5) was found for $\omega(k) \neq ck + \gamma$. However, in the example considered, only the velocity of the right-hand wave front is finite, while the velocity of the left-hand front is infinite. But since in [1] physical meaning was assigned only to the region $z \geq 0$, this fact remained unnoticed.

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Forestry Academy
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Leningrad

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Note: Figure translations are in progress. See original paper for figures.