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Abstract

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MATHEMATICS

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ON NONDEGENERATE SUBELLIPTIC PSEUDODIFFERENTIAL OPERATORS

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In this note conditions are formulated on the symbol $p^0(x,\xi)$ of a pseudodifferential operator P of order m, under which the estimate

$$|u|_{s-(k-1)/k} \leq C(K)(|Pu|_{s-m} + |u|_{s-1}), \qquad u \in C_0^\infty(K), \tag{1}$$

holds, where $|u|_s$ is the norm of the function u in the L. N. Slobodetskii space H^s, k is a natural number, and K is a compact set in $\Omega \subset \mathbb{R}^n$.

These conditions are exact, necessary and sufficient, if the operator is nondegenerate. Here we call an operator P nondegenerate if its vector

$$\nabla_{x,\xi} p^0(x,\xi) \equiv (\partial p^0/\partial x_1, \dots, \partial p^0/\partial x_n, \partial p^0/\partial \xi_1, \dots, \partial p^0/\partial \xi_n)$$

is not proportional to a real vector at each characteristic point $(x,\xi) \in K \times S^{n-1}$, i.e. at a point where $p^0(x,\xi) = 0$.

From the conditions obtained it follows that if, for an operator P, the estimate

$$|u|_{s-1+\delta} \leq C_1(K)(|Pu|_{s-m} + |u|_{s-1}), \qquad u \in C_0^\infty(K), \tag{2}$$

is valid and $1/(k+1) < \delta \leq 1/k$, then there exists a constant $C = C(K)$ such that for all functions $u \in C_0^\infty(K)$ inequality (1) is valid.

Such results were obtained earlier by L. Hörmander in ([1]) for the case $k = 1$ and by the author in ([4,5]) for the cases $k = 2$ and $k = 3$.

Without loss of generality, in what follows one may assume that $m = 1$, and that P is an operator mapping functions from $C_0^\infty(K)$ into $C^\infty(K')$, where K' is a compact set containing K in its interior (see ([1])).

Let $P = P_1 + iP_2 + P_3$, where P_1 and P_2 are first-order operators with positively homogeneous in ξ real symbols $p_1(x,\xi)$ and $p_2(x,\xi)$, respectively, and P_3 is an operator of order zero. The nondegeneracy condition means that the $2n$-dimensional vectors $\nabla_{x,\xi} p_1(x,\xi)$ and $\nabla_{x,\xi} p_2(x,\xi)$ are not collinear if $p^0(x,\xi) = 0$.

Denote by C the first-order pseudodifferential operator equal to $i[P_2, P_1] = i(P_2P_1 - P_1P_2)$, and by $c(x,\xi)$ the principal part of the symbol of this operator. As L. Hörmander showed in ([1]), if $c(x,\xi) < 0$ at a characteristic point (x,ξ), then estimate (2) is impossible for the operator P for any real δ. If, however, $c(x,\xi) > 0$ at all characteristic points (x,ξ), then estimate (2) holds for $\delta = 1/2$. Suppose that at some point $(x,\xi) \in K \times S^{n-1}$ we have $p^0(x,\xi) = c(x,\xi) = 0$. Consider at this point the values of the principal parts of the symbols of the operators which are successive commutators of the operators P_1 and P_2. Let the first (with minimal s) nonzero value correspond to the commutator $[\dots[[P_{j_1}, P_{j_2}], P_{j_3}], \dots, P_{j_s}]$. Then we set $k(x,\xi) = s - 1$. In this way we define the function $k(x,\xi)$ at all characteristic points $(x,\xi) \in K \times S^{n-1}$ of the operator P.

We shall now formulate the main results of this work.

Theorem 1. Estimate (1) is valid for a nondegenerate operator P if and only if:

1) $c(x,\xi) \geqslant 0$ at those points $(x,\xi) \in K \times S^{n-1}$ where $p^0(x,\xi) = 0$;

2) $k(x,\xi)$ assumes odd values not exceeding $k + 1$.

Theorem 2. If for a nondegenerate operator P estimate (2) holds and $1/2(l+1) < \delta \leqslant 1/2l$, then there exists a constant $C_2 = C_2(K)$ such that

$$|u|_{s-1+1/2l} \leqslant C_2(K)\left(|Pu|_{s-m} + |u|_{s-1}\right)$$

for all functions $u \in C_0^\infty(K)$.

It is not difficult to see that Theorem 2 follows directly from Theorem 1.

Let us note some consequences of Theorem 1. As is easy to show (see ([5])), the set of points x, ξ for which $p^0(x,\xi) = 0$ forms, in the case of a nondegenerate operator, a manifold of dimension $2n-2$. If for the operator P estimate (2) holds with some $\delta > 0$, then $c(x,\xi)$ cannot have a zero of infinite order at a characteristic point. Therefore the set of points (x,ξ) for which $p^0(x,\xi) = c(x,\xi) = 0$ has dimension $\leqslant 2n-3$. The example of an operator P with symbol

$$p(x,\xi) = i\xi_1 + \xi_2 - x_1(\xi_1^2 + \dots + \xi_l^2)|\xi|^{-1} - x_1(x_1^2 + \dots + x_m^2)|\xi|$$

$$(1 \leqslant m \leqslant n, \qquad 2 \leqslant l \leqslant n-1),$$

for which, by Theorem 1, estimate (1) holds with $k = 4$, shows that the dimension of the set $p^0(x,\xi) = c(x,\xi) = 0$ can assume any values between 1 and $2n-3$.

Another example of an operator satisfying the conditions of Theorem 1 was noted in ([6]). This is the operator with symbol

$$p(x,\xi) = i\xi_1 + \xi_2 + a x_1^l |\xi|,$$

where l is odd and $a < 0$. It is easy to verify that for this operator estimate (2) is fulfilled with $\delta = 1/(l+1)$.

The proof of Theorem 1 uses the conditions, obtained in ([2,4]), of explicit form, necessary and sufficient for the validity of inequality (1). The necessity of condition 1) was proved by L. Hörmander in ([1]). The necessity of condition 2) can be proved with the aid of the results of our work ([6]), in which conditions were obtained that are necessary and sufficient for the validity of inequalities of the form

$$\int |u|^2 e^{Q(x,\mu)}\,dx \leqslant C \int \left|\sum_{j=1}^{n} a_j \frac{\partial u}{\partial x_j}\right|^2 e^{Q(x,\mu)}\,dx, \qquad u \in C_0^\infty(\mathbb{R}^n),$$

where a_j are complex numbers; $Q(x,\mu)$ is a homogeneous polynomial in $x \in \mathbb{R}^n$, $\mu \in \mathbb{R}^r$, with $\mu_j \geqslant 0,\ j = 1, \dots, r$. From this latter work, as well as from ([5]), it is clear why the conditions of Theorem 1 are not necessary in the general case.

For the proof of the sufficiency of the conditions of Theorem 1, the results obtained in ([2,4]), as well as the methods developed by L. Hörmander in ([3]) in connection with the study of hypoelliptic equations of second order, are used. These methods are connected with the use of certain facts from the theory of noncommutative Lie algebras.

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