INTEGRAL REPRESENTATIONS OF HOLOMORPHIC FUNCTIONS AND TAYLOR’ S FORMULA

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Abstract

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INTEGRAL REPRESENTATIONS OF HOLOMORPHIC FUNCTIONS AND TAYLOR' S FORMULA

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M. M. Dzhrbashyan (1), for functions analytic in a disk, by introducing integro-differential operators $D^{-\alpha}$ of arbitrary order α $(-1 < \alpha < +\infty)$ in the sense of Riemann-Liouville, obtained a generalization of the Cauchy and Schwarz integral formulas that is of great importance in the essential generalization of the apparatus of representations of meromorphic functions and of the basic propositions of the Nevanlinna theory of these functions. In the present note, the indicated generalization of the Cauchy and Schwarz integral formulas is used (§ 1*) for the corresponding generalization of the corresponding integral formulas in the case of bounded convex complete circular domains of the space C^n, n $(n \geq 2)$, of complex variables. In addition, in § 2 a simpler form is given for writing the remainder term, established earlier by the author (5) in the case of n $(n \geq 1)$ complex variables, in Taylor' s formula.

We shall also point out the following proposition of M. M. Dzhrbashyan (1, p. 594), needed in § 1:

If the function

$$f(re^{i\varphi}) = \sum_{k=0}^{\infty} a_k (re^{i\varphi})^k$$

is holomorphic in the disk $|z| < R$, then for any α $(-1 < \alpha < +\infty)$ the function

$$f_\alpha(re^{i\varphi}) \equiv r^{-\alpha} D^{-\alpha} f(re^{i\varphi}) = \sum_{k=0}^{\infty} a_k \frac{\Gamma(1+k)}{\Gamma(1+\alpha+k)} (re^{i\varphi})^k$$

is holomorphic in the same disk $|z| < R$.

§ 1. **Theorem.** *Let* $D \equiv (T)$, *the function* $f(z)$ $(z = (z_1, \ldots, z_n),\ n \geq 2)$ *be holomorphic in* D, *and let* q *be a number equal to 0 or 1. Then for* $k = 0, 1, 2, \ldots;\ \tilde{k} = 0, 1, 2, \ldots,$ *arbitrary* α $(-1 < \alpha < +\infty)$, *and* ρ $(0 < \rho < 1)$

$$
\begin{aligned}
f(z) = qf(0) + \frac{1}{n+q(1-n)} \times \\
\times \sum_{\nu=1}^{n} \frac{z_\nu^q}{(2\pi)^n} \int d\omega_\zeta \int d\omega_\theta \int_{-\pi}^{\pi} L_{q+1,n-1}^{(n-q-1)} \left[L_{A\widetilde{A}}^{(k,\tilde{k})} \left[C_\alpha \left(e^{-i\varphi} \frac{u}{\rho} \right) \right] \right] \times \qquad (1) \\
\times \rho^{-\alpha} D^{-\alpha} L_{A\widetilde{A}}^{(k,-\tilde{k})} \left[F_{0\nu}^{(q)} (\rho e^{i\varphi}, r, \theta) \right] \, d\varphi \, {*}{*},
\end{aligned}
$$

where

$$
D_\rho = \rho D, \qquad C_\alpha(w) = \Gamma(1+\alpha)/(1-w)^{1+\alpha} \ (|w| < 1) \, {*}{*}{*} \, .
$$

* In § 1, in addition to the notation from (^{1}) used in this article, the former notation (as well as definitions) from ($^{2-10}$) is also used.

** The operator $D^{-\alpha}$ acts on the operator

$$
L_{A\widetilde{A}}^{(k,-\tilde{k})} \left[F_{0\nu}^{(q)} (\rho e^{i\varphi}, r, \theta) \right]
$$

as on a function of ρ.

*** By $(1-w)^{1+\alpha}$ we mean $e^{(1+\alpha)\ln(1-w)}$, where under the logarithm is understood the branch that equals 0 at $w = 0$.

Under the conditions of this theorem, for $k = 0, 1, 2, \ldots$; $\tilde{k} = 0, 1, 2, \ldots$, arbitrary α $(-1 < \alpha < +\infty)$ and ρ $(0 < \rho < 1)$, the following formula also holds:

$$
\begin{aligned}
\underset{z \in D_\rho}{f(z)} = qf(0) + \frac{i}{n+q(1-n)} \sum_{\nu=1}^{n} z_\nu^q \, \mathrm{Im} \, f_\nu^{(q)}(0) + \\
+ \frac{1}{n+q(1-n)} \sum_{\nu=1}^{n} \frac{z_\nu^q}{(2\pi)^n} \int d\omega_\tau \int d\omega_\theta \int_{-\pi}^{\pi} L_{q+1,n-1}^{(n-1-q)} \left[L_{A\tilde{A}}^{(-k,\tilde{k})} \left[S_\alpha \left(e^{-i\varphi} \frac{u}{\rho} \right) \right] \right] \times \\
\times \mathrm{Re} \, \rho^{-\alpha} D^{-\alpha} L_{A\tilde{A}}^{(k,-\tilde{k})} \left[F_{0_\nu}^{(q)} (\rho e^{i\varphi}, r, \theta) \right] \, d\varphi,
\end{aligned}
\qquad (2)
$$

where

$$
S_\alpha(w) = 2C_\alpha(w) - C_\alpha(0) = \Gamma(1+\alpha) \left\{ \frac{2}{(1-z)^{1+\alpha}} - 1 \right\} \quad (|w| < 1).
$$

In the course of proving the integral formulas (1), (2), one uses the integral formulas of M. M. Dzhrbashyan (^{1}), p. 594, formulas (2.5), (2.6), which are the above-mentioned generalization of the Cauchy and Schwarz integral formulas, and integral formula (4) from the author’ s paper (^{9}).

Among the large number of consequences following from formulas (1), (2), we point out the following two ($q = 0,\ \tilde{k} = 0,\ k = n-1,\ A_j = (j, 1, \dots, 1),\ j = 1, \dots, k$):

$$1.\quad \underset{z\in D_\rho}{f(z)} = \frac{1}{(2\pi)^n} \int d\omega_\tau \int d\omega_\theta \int_{-\pi}^{\pi} C_\alpha \left(e^{-i\varphi}\frac{u}{\rho}\right) \rho^{-\alpha} D^{-\alpha} L_{1,n-1}^{(n-1)} \left[F_0(\rho e^{i\varphi}, r, \theta)\right] d\varphi.$$

$$2.\quad \underset{z\in D_\rho}{f(z)} = i\,\mathrm{Im}\, f(0) + \frac{1}{(2\pi)^n} \int d\omega_\tau \int d\omega_\theta \int_{-\pi}^{\pi} S_\alpha \left(e^{-i\varphi}\frac{u}{\rho}\right) \times \mathrm{Re}\, \rho^{-\alpha} D^{-\alpha} L_{1,n-1}^{(n-1)} \left[F_0(\rho e^{i\varphi}, r, \theta)\right] d\varphi.$$

§ 2. In the case of n ($n \geqslant 1$) complex variables, the author (^{5}) obtained a Taylor formula in which the remainder term is determined by the formula

$$R_m(z_1, \dots, z_n) = \frac{1}{(m-1)!} \int_0^1 d\varepsilon \int_0^\varepsilon (\varepsilon - t)^{m-1} \left(\sum_{\nu=1}^{n} (z_\nu - z_\nu^0) \frac{\partial}{\partial_\nu}\right)^{m+1} f(Z_1, \dots, Z_n)\, dt$$

$$(Z_1 = z_1^0 + t(z_1 - z_1^0), \dots,\ Z_n = z_n^0 + t(z_n - z_n^0),\ \varepsilon \text{ and } t \text{ real}*).$$

Let us note, however, that the remainder term $R_m(z_1, \dots, z_n)$ can be written in the simpler form

$$R_m(z_1, \dots, z_n) = \frac{1}{m!} \int_0^1 (1 - \varepsilon)^m \left(\sum_{\nu=1}^{n} (z_\nu - z_\nu^0) \frac{\partial}{\partial_\nu}\right)^{m+1} f(Z_1, \dots, Z_n)\, d\varepsilon$$

$$(Z_1 = z_1^0 + \varepsilon(z_1 - z_1^0), \dots,\ Z_n = z_n^0 + \varepsilon(z_n - z_n^0)).$$

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* For the remaining explanations, see ([5]).

Note: Figure translations are in progress. See original paper for figures.

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