ON THE REGULAR GLOBAL REALIZATION IN E^3 OF TWO-DIMENSIONAL METRICS OF CLASS C^2 WITH NEGATIVE CURVATURE OF CLASS C^1

1969

SovietRxiv

Abstract

Full Text

UDC 513.73

MATHEMATICS

E. V. SHIKIN

ON THE REGULAR GLOBAL REALIZATION IN E^3 OF TWO-DIMENSIONAL METRICS OF CLASS C^2 WITH NEGATIVE CURVATURE OF CLASS C^1

(Presented by Academician P. S. Aleksandrov, 28 V 1969)

1. The problem of the global realization of two-dimensional metrics W of negative curvature was first formulated by E. G. Poznyak (see (^{1})). In paper (^{1}) a theorem was proved on the regular global realization of such metrics under the condition that $W \in \overline{C}^{4,1}*$ and the curvature K is contained between two negative constants $-k_1^2 \leq K \leq -k_2^2 < 0$. In the proof of this theorem the basic equations of the theory of surfaces of negative curvature in Riemannian invariants were used. Using the method developed in paper (^{1}), and applying the existence theorem of Hartman and Wintner (see (^{2})), one can prove an analogous theorem on the global realization of two-dimensional metrics W of negative curvature under weakened regularity requirements: $W \in \overline{C}^4$ and $-k_1^2 \leq K \leq -k_2^2 < 0$. With a further weakening of the regularity conditions this method, apparently, cannot be used to clarify the question of the possibility of the global realization of metrics of negative curvature.

In the present work the question is considered of the global realization in E^3 of two-dimensional metrics W of class C^2 with negative curvature of class C^1. A new method is used here, based on the application of the Darboux equation.

2. Let a metric W of negative curvature K be given in the infinite strip $\Pi_a = \{0 \leq x \leq a,\ -\infty < y < +\infty\}$ by means of the line element

$$ds^2 = dx^2 + B^2(x,y)\, dy^2 \tag{1}$$

and let the following regularity conditions be satisfied: $B \in \overline{C}^2$ in Π_a, $K \in \overline{C}^1$, $-k_1^2 \leq K \leq -k_2^2 < 0$, and $\operatorname{grad} K$ is uniformly continuous in Π_a.

Remark 1. If the metric W is given in Π_a by means of the line element (1) and satisfies the regularity conditions, then it can be extended to the whole plane so that it will be given on this plane by a line element of the form (1), with preservation of the regularity conditions in any strip containing Π_a.

Theorem 1. *Under the regularity conditions formulated above, the metric W in the strip Π_a can be realized in E^3 by means of a surface of class C^2.*

From Theorem 1 the following theorem follows.

Theorem 2. *Let W be a complete two-dimensional metric of class C^2 with negative curvature of class C^1. Then the metric in any geodesic disk of the metric W can be realized in E^3 by means of a surface of class C^2.*

Indeed, any such disk can be placed in a strip satisfying the conditions of Theorem 1.

3. In this section we shall report some information about the Darboux equation. Let $\{X(x,y), Y(x,y), Z(x,y)\}$ be the parametric equations of a surf
–

ness. Then the coordinate Z satisfies the following Darboux equation (see (3)):

$$A_1 + A_2 r + A_3 s + A_4 t + A_5(rt - s^2) = 0, \tag{2}$$

where $p = Z_x,\ q = Z_y,\ r = Z_{xx},\ s = Z_{xy},\ t = Z_{yy}$, and A_i $(i = 1,2,3,4,5)$ are known functions of the metric, Z, p, and q. The converse assertion is also valid: if Z is a solution of equation (2) in a neighborhood of some point, then, from the function Z, under additional restrictions one can determine functions X, Y so that $\{X(x,y), Y(x,y), Z(x,y)\}$ are parametric equations of a surface realizing the prescribed metric (see, for example, (4)).

Equation (2) is reduced to a system of 5 quasilinear first-order equations in the unknowns x, y, Z, p, q, with characteristic coordinates u and v as independent variables (see (5)). For the metric (1) this system can be put in the form:

$$x_{uv} = -(\ln k)_x x_u x_v - (\ln\sqrt{k})_y (x_u y_v + x_v y_u) + B B_x y_u y_v, \tag{3}$$

$$y_{uv} = -(\ln\sqrt{kB})_x (x_u y_v + x_v y_u) - (\ln kB)_y y_u y_v; \tag{4}$$

$$\Sigma_{uv} = \Phi,$$

where $k = \sqrt{-K}$, Σ is the vector with coordinates Z, p, q; Φ is the vector with coordinates Φ_1, Φ_2, Φ_3, and the functions Φ_i are expressed linearly through

---

* The symbol $C^{n,1}$ is explained in paper (1).

$p, q, \sqrt{B^2(1-p^2)-q^2}$ with coefficients depending on x, y and their first derivatives with respect to u and v.

Remark 2. In the case of an arbitrary line element the right-hand sides of the equations of system (3) depend on Z, p, and q. The absence of these functions in system (3) for the line element (1) plays an essential role in the subsequent reasoning.

4. Let us outline the main stages of the proof of Theorem 1. We shall assume that the metric W is given in the xoy-plane and satisfies the conditions of item 2. Since the functions Z, p, q do not enter into the right-hand sides of the equations of system (3), this system can be studied separately from the whole system (3)–(4).

Let Γ be the line in the parameter plane u, v defined by the equation $u + v = 0$, and let Γ^* be any fixed line parallel to it. By Π^* we shall denote the strip of the (u, v)-plane bounded by the lines Γ and Γ^*. We shall solve system (3) with the following initial data:

$$x|_\Gamma = 0; \qquad y|_\Gamma = 2\varepsilon u; \qquad x_u|_\Gamma = \frac{1}{k}; \qquad x_v|_\Gamma = \frac{1}{k}; \qquad y_u|_\Gamma = \varepsilon; \qquad y_v|_\Gamma = -\varepsilon. \tag{5}$$

The following assertion on the existence in the large of a solution of system (3) is valid.

Lemma. *For sufficiently small ε, in the strip Π^* there exists a solution of system (3) of class C^*, satisfying the initial data (5). Moreover the derivatives of the solution with respect to u, v have the form:*

$$x_u = \frac{1}{k} + O(\varepsilon); \qquad x_v = \frac{1}{k} + O(\varepsilon); \qquad y_u = \varepsilon + O(\varepsilon^2); \qquad y_v = -\varepsilon + O(\varepsilon^2). \tag{6}$$

The proof of the lemma is carried out by the method of successive approximations, with essential use of the structure of the right-hand sides of system (3).

Remark 3. The lemma formulated is valid not only for initial data of the form (5). In fact, small changes in the initial data lead to the same result.

Remark 4. We note that for all sufficiently small ε the Jacobian of the solution $J = x_u y_v - x_v y_u$ is nonzero in Π^*.

* A function $f(x, y)$ belongs to the class C^* if it belongs to the class C^1 and has a continuous mixed derivative f_{xy}.

Remark 5. If the strip Π^* is chosen sufficiently wide and ε sufficiently small, then from formula (6) and the inequality on the Jacobi field J it follows that, by means of the functions $x(u,v)$ and $y(u,v)$, a one-to-one mapping of the strip Π^* onto a domain in the xoy-plane is carried out, which covers Π_a. The images of the lines $u = \text{const}$ and $v = \text{const}$, which are characteristics in the xoy-plane, form a regular net regularly covering Π_a. In this case the image of the initial line Γ will be the oy-axis.

Let $(0, y_0)$ be an arbitrary point of the oy-axis. Then, using the lemma and system (4), one can prove that the metric W in the square $\Delta : -\varepsilon^2 \le x \le \varepsilon^2,\ y_0 - \varepsilon^2 \le y \le y_0 + \varepsilon^2$ is realized by means of a surface S of class C^2 in such a way that the Z-axis is the normal to S at the point $(0, y_0)$. Hence follows the possibility of realizing the metric W in Π_{ε^2}. If, furthermore, the line $x = \varepsilon^2/2$ is taken as the image of the initial curve, then one can verify the possibility of realizing the metric W in $\Pi_{3/2\varepsilon^2}$.

Continuing these arguments, we exhaust the entire strip Π_a.

The author is sincerely grateful to his teacher E. G. Poznyak and to all participants of the geometry seminar in general for their attention to this work.

Moscow Engineering Physics
Institute

Received
28 V 1969

REFERENCES

1. E. G. Poznyak, DAN, 170, No. 4, 786 (1966).
2. P. Hartman, A. Wintner, *Am. J. Math.*, 74, 4, 834 (1952).
3. V. F. Kagan, *Foundations of the Theory of Surfaces*, 1,
4. R. Courant, *Partial Differential Equations*, Moscow, 1964.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.