ON ONE METHOD FOR SOLVING THE BOLTZMANN EQUATION IN THE SPATIALLY HOMOGENEOUS CASE

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Abstract

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ON ONE METHOD FOR SOLVING THE BOLTZMANN EQUATION IN THE SPATIALLY HOMOGENEOUS CASE

(Presented by Academician V. V. Struminskii, 26 IV 1968)

In the work of V. V. Struminskii (^{1}) a fairly general method was developed for solving the Boltzmann equation, based on expanding the sought distribution function and the time derivatives of its first five moments in a series in a small parameter. In the recurrent system of integro-differential equations in this case the first approximation will be the spatially homogeneous Boltzmann equation. This equation was first investigated in detail by T. Carleman (^{2}). He proved the existence and uniqueness of the solution of the equation for the case when the distribution function depends on the modulus of the velocity $r = (v_x^2 + v_y^2 + v_z^2)^{1/2}$, and its initial value is bounded, $0 \leq f_0(r) \leq a/(\sqrt{1+r^2})^{\varkappa},\ \varkappa > 6$. In addition, he showed that the solution tends to the Maxwell distribution as $t \to \infty$.

In the present paper we consider the solution of the spatially homogeneous Boltzmann equation for a sufficiently general form of initial distribution functions, represented in the form of a series of Hermite polynomials.

The Boltzmann equation in the spatially homogeneous case has the form

$$\frac{\partial g}{\partial t} = \frac{\rho}{M} \iint (g' g_1' - g g_1) B(\vartheta, V)\, d\vartheta\, d\varepsilon\, dv_1, \tag{1}$$

where $g = \frac{(RT)^{3/2}}{\rho} f$ is the dimensionless distribution function; $v = c/\sqrt{RT}$ is the dimensionless velocity. The solution is sought in the form of a series of Hermite polynomials

$$g(v, t) = g^0(v) \sum_{n=0}^{\infty} \frac{1}{n!} a_i^n(t) H_i^n(v) \tag{2}$$

with tensor coefficients

$$a_i^n(t) = \int g(v,t) H_i^n(v)\, dv, \tag{3}$$

where $g^0(v)$ is the Maxwellian distribution function.

We multiply both sides of equation (1) by $H_i^n(v)$ and integrate with respect to v. Then, taking into account (3) and the integral symmetry relations, equation (1) is rewritten in the form

$$\frac{\partial a_i^n}{\partial t} = \frac{\rho}{2M} \int g(v) g(v_1) (H_i^n(v') - H_i^n(v)) B(\vartheta, V)\, d\vartheta\, d\varepsilon\, dv\, dv_1. \tag{4}$$

Under the assumption of the quasi-Maxwellian model of interaction, when the scattering cross section does not depend on the relative velocity of the colliding molecules and the radius of action of the molecular forces is finite, equation (4) can be written in the form

$$\frac{\partial a_i^n}{\partial t} = \frac{4\pi}{\tau} \iint g(v) g(v_1) (H_i^n(v') - H_i^n(v)) \vartheta\, d\vartheta\, d\varepsilon\, dv\, dv_1, \tag{5}$$

where τ is the mean free time of Maxwell molecules.

It can be shown that in the case of Maxwell molecules (^{3}), after integrating the right-hand side of (5), for the coefficients $a_i^n(t)$ one obtains the re-

a current system of ordinary differential equations

$$\frac{\partial a_i^n}{\partial t} = \sum_{r+s=n} \beta_{ijk}^{n,r,s} a_j^r a_k^s. \tag{6}$$

The main problem consists in finding the coefficients $\beta_{ijk}^{n,r,s}$ for an arbitrary tensor number n. We introduce for this purpose a Cartesian coordinate system such that the relative velocity of two colliding molecules has coordinates $V(V,0,0)$, and the direction of the line of impact is $\alpha(\cos\vartheta, \sin\vartheta\cos\varepsilon, \sin\vartheta\sin\varepsilon)$. In equation (5), instead of the difference $(H_i^n(v') - H_i^n(v))$, we shall consider the difference $({v'}^n - v_i^n)$, since only these first terms of the Hermite polynomials affect the form of $\beta_{ijk}^{n,r,s}$. From the dynamics of collisions it is known that $v' = v + \alpha|V|\cos\vartheta$. Then

$$\int_0^{2\pi} ({v'}^n - v^n)\, d\varepsilon = \int_0^{2\pi} \{(v + \alpha|V|\cos\vartheta)^n - v^n\}\, d\varepsilon = \sum_{\Lambda=1}^{n} v^{n-\Lambda} |V|^\Lambda \cos^\Lambda\vartheta \int_0^{2\pi} \alpha^\Lambda\, d\varepsilon. \tag{7}$$

In the chosen coordinate system the integrals

$$|V|^{\Lambda} \cos^{\Lambda} \vartheta \int_0^{2\pi} \alpha^{\Lambda} \, d\varepsilon$$

can be expressed through symmetric tensors of the form $V^{\Lambda-2k}\delta^{k}$. By virtue of the invariance of the tensors $V^{\Lambda-2k}$ and δ^{k} with respect to the full orthogonal transformation group, including rotation, the form of these integrals is preserved in any other coordinate system rotated relative to the chosen one by an arbitrary angle. We shall have for any Λ:

1) for even Λ

$$|V|^{\Lambda} \cos^{\Lambda} \vartheta \int_0^{2\pi} \alpha^{\Lambda} \, d\varepsilon =$$

$$= \sum_{k=0}^{\Lambda/2} \left(\sum_{p=0}^{k} (-1)^{k+p} \frac{2\pi \sin^{\Lambda-2p} \vartheta \, \cos^{2p} \vartheta \, (2k)!}{\left(\frac{\Lambda-2p}{2}\right)! \, 2^{(\Lambda-2p)/2} 2^{(2k-2p)/2} \left(\frac{2k-2p}{2}\right)! (2p)!} \right) \times$$

$$\times |V|^{\Lambda-2k} V^{2k} \delta^{(\Lambda-2k)/2} \cos^{\Lambda} \vartheta, \tag{8}$$

2) for odd Λ

$$|V|^{\Lambda} \cos^{\Lambda} \vartheta \int_0^{2\pi} \alpha^{\Lambda} \, d\varepsilon =$$

$$= \sum_{k=0}^{[\Lambda/2]} \left(\sum_{p=0}^{k} (-1)^{k+p} \frac{2\pi \sin^{\Lambda-(2p+1)} \vartheta \, \cos^{2p+1} \vartheta \, (2k+1)!}{\left(\frac{\Lambda-(2p+1)}{2}\right)! \, 2^{[\Lambda-(2p+1)]/2} 2^{(2k-2p)/2} \left(\frac{2k-2p}{2}\right)! (2p+1)!} \right) \times$$

$$\times |V|^{\Lambda-(2k+1)} V^{2k+1} \delta^{(\Lambda-2k+1)/2} \cos^{\Lambda} \vartheta.$$

After simple algebraic transformations and integration with respect to v and v_1, equation (5) takes the form:

$$\frac{\partial a_i^n}{\partial t} = \frac{2\pi\rho}{M} \sum_{m=0}^{[n/2]} \left\{ \sum_{l=0}^{n-2m} \left[(n-2m-l)! \, \psi(l,m,n) \times \right. \right.$$

$$\times \left(\sum_{k=0}^{m} \sum_{s=0}^{\min[k,m-k]} C_m^{k-s} 2^{2s} C_{m-k-s}^{2s} a_{i(2m-2k,\, 2s)}^{l+2m-2k} a_{(2k,\, 2s)}^{n-2m-l+2k}\right.$$

$$-\sum_{k=0}^{m-1}\ \sum_{s=0}^{\min(k,m-k)} C_m^{k-s} 2^{2s+1} C_{m-k+s}^{2s+1} a_{i(2m-2k-1,\,2s+1)}^{l+2m-2k-1} a_{(2k+1,\,2s+1)}^{n-l-2m+2k+1} \Bigg) \ ] \Bigg\} \delta^m,$$

where

$$\psi(l,m,n) = \int_0^{\pi/2} B(\vartheta) \Bigg[\sum_{r=[(l+1)/2]+m}^{[n/2]} \Bigg(\sum_{p=0}^{r-m} (-1)^{r-m+p} \frac{\sin^{2r-2p}\vartheta \cdot \cos^{2r+2p}\vartheta\,(2r-2m)!}{(r-p)!\,2^{2r-2p-m}(r-p-m)!\,(2p)!} \times (-1)^l \frac{1}{(n-2r)!\,(2r-2m-l)!} \Bigg) + \sum_{r=[l/2]+m}^{[n/2]-1} \Bigg(\sum_{p=0}^{r-m} (-1)^{r-m+p} \frac{\sin^{2r-2p}\vartheta \cdot \cos^{2r+2p+2}\vartheta\,(2r-2m+1)!}{(r-p)!\,2^{2r-2p-m}(r-p-m)!\,(2p+1)!} \times (-1)^{l-1} \frac{1}{(n-2r-1)!\,(2r-2m-l+1)!} \Bigg) \Bigg] d\vartheta. \tag{9}$$

This recurrent system of equations for the coefficients $a_i^n(t)$ makes it possible in the general case to construct the solution of equation (1) by successively computing the coefficients $a_i^n(t)$ from (9) and using the initial conditions. In the general case, the study of convergence of the solution is connected with considerable difficulties and will be given in a special paper. However, in the case of spherical symmetry these investigations can be carried out comparatively simply. In this case equation (9) takes the form

$$\frac{\partial a_i^n}{\partial t} = \Lambda_n a^n + \sum_{m=2}^{[n/4]} C_{n/2}^m D_m a^{n-2m} a^{2m}, \tag{10}$$

where

$$\Lambda_n = \frac{2\pi\rho}{M} \int_0^{\pi/2} (-1 + \sin^n\vartheta + \cos^n\vartheta)\, B(\vartheta)\, d\vartheta \qquad (n \text{ even}),$$

$$D_m = \frac{2\pi\rho}{M} \int_0^{\pi/2} \sin^{2m}\vartheta \cos^{2m}\vartheta \left(\sin^{n-4m}\vartheta + \cos^{n-4m}\vartheta - \delta_{m,n/4} \right) B(\vartheta)\, d\vartheta.$$

In computing these integrals, following (4), one may take $B(\vartheta) = 4\vartheta/\pi^2$. Then equation (10) can be written in the form

$$\frac{\partial a^n}{\partial t} = \Lambda_n a^n + \frac{2\pi\rho}{M} \sum_{m=2}^{[n/4]} \frac{(n-2m)!\,(2m)!}{2^n\{[(n-2m)/2]!\}^2(m!)^2}\, a^{n-2m} a^{2m}.$$

Integrating, we obtain

$$a^n(t) = a^n(0)e^{\Lambda_n t} + \frac{2\pi\rho}{M} e^{\Lambda_n t} \sum_{m=2}^{[n/4]} \frac{(n-2m)!\,(2m)!}{2^n\{[(n-2m)/2]!\}^2(m!)^2} \int_0^t e^{-\Lambda_n \tau} a^{n-2m}(\tau) a^{2m}(\tau)\, d\tau \tag{11}$$

or

$$|a^n(t)| \leq |a^n(0)|e^{\Lambda_n t} + \frac{2\pi\rho}{M} e^{\Lambda_n t} \sum_{m=2}^{[n/4]} \frac{(n-2m)!\,(2m)!}{2^n\{[(n-2m)/2]!\}^2(m!)^2} \int_0^t e^{-\Lambda_n \tau} |a^{n-2m}(\tau)|\, |a^{2m}(\tau)|\, d\tau. \tag{12}$$

To investigate the convergence of series (2), we shall estimate the coefficients $|a^n(t)|$ by the method of induction. First assume that the initial values of the expansion coefficients satisfy the inequalities

$$|a^n(0)| < d_n. \tag{13}$$

We shall show that, if for $k < n$

$$|a^k(t)| < d_k, \tag{14}$$

then for $k = n$ condition (14) will be satisfied. Substituting (14) and (13) into (12), we obtain

$$|a^n(t)| < d_n + (1 - e^{\Lambda_n t}) \left\{ \frac{2\pi\rho}{M} \sum_{m=2}^{[n/4]} \frac{(n-2m)!(2m)!\, d_{n-2m} d_{2m}}{2^n\{[(n-2m)/2]!\}^2(m!)^2} - d_n \right\}.$$

Our assertion will be proved for a fairly broad class of initial distribution functions satisfying the condition

$$d_n \geq \frac{2\pi\rho}{M} \sum_{m=2}^{[n/4]} \frac{(n-2m)!(2m)!\, d_{n-2m} d_{2m}}{2^n\{[(n-2m)/2]!\}^2(m!)^2}. \tag{15}$$

In the particular case, for example, (15) is satisfied for the quantities $d_n = \sqrt{(n-2)!}$. In this case

$$\sum_{n=4}^{\infty} \frac{|a^n(t)|^2}{n!} < \sum_{n=4}^{\infty} \frac{d_n^2}{n!} = \sum_{n=1}^{\infty} \frac{1}{n(n+1)},$$

and the series (2) converges in norm.

Let us investigate the rate at which an arbitrary initial distribution approaches the Maxwellian distribution. Let us estimate the character of the change of the coefficient $a^n(t)$. Assume that for $k < n$ the conditions

$$a^k(t) = c_k(t)\exp(\Lambda_k t), \quad \text{where } |c_k(t)| < b_k, \tag{16}$$

are satisfied, and show that this condition is satisfied also for $k = n$. Substituting (16) into (11), we obtain $a^n(t) = c_n(t)\exp(\Lambda_n t)$, where

$$c_n(t) = a^n(0) +$$

$$+ \frac{2\pi\rho}{M} \sum_{m=2}^{[n/4]} \frac{(n-2m)!(2m)!}{2^n\{[(n-2m)/2]!\}^2(m!)^2} \int_0^t e^{(\Lambda_{n-2m}+\Lambda_{2m}-\Lambda_n)\tau} c_{n-2m}(\tau)c_{2m}(\tau)\,d\tau. \tag{17}$$

Using the expression given above for Λ_n and carrying out uncomplicated trigonometric calculations, it is not difficult to show that $\Lambda_{n-2m} + \Lambda_{2m} - \Lambda_n = \varepsilon_{2m}^n < 0$. In this case, from (17) we shall have

$$|c_n(t)| \le |a^n(0)| + \frac{2\pi\rho}{M} \sum_{m=2}^{[n/4]} \frac{(n-2m)!(2m)!}{2^n\{[(n-2m)/2]!\}^2(m!)^2|\varepsilon_{2m}^n|} (1 - e^{\varepsilon_{2m}^n t})\, b_{n-2m} b_{2m},$$

which proves our initial assertion. Then

$$f - f^0 = f^0 \sum_{n=1}^{\infty} \frac{c_n(t)e^{\Lambda_n t}}{n!} H^n(v),$$

where the $c_n(t)$ are bounded in time. It follows from this expression that an arbitrary initial distribution obeying condition (15) will rapidly lose its initial information, tending to the Maxwellian distribution.

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Note: Figure translations are in progress. See original paper for figures.

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