ON MODELING THE PATHS OF TEST BODIES IN A GRAVITATIONAL FIELD

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Abstract

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The principal method of investigating a gravitational field consists in studying the behavior of test bodies in this field. For Einstein' s theory of gravitation, fundamental in this sense is the principle of geodesics, according to which test bodies in a gravitational field move along geodesic lines (nonisotropic for bodies or particles with nonzero rest mass; isotropic in the opposite case); as was shown simultaneously and independently by Einstein, Infeld, and Hoffmann, on the one hand ([1]), and by Fock ([2,3]), on the other, this principle may be regarded as a consequence of the field equations under certain assumptions about the structure of the mass tensor. Choosing a canonical parameter along the paths, the equations of motion of test bodies can always be represented in the form

$$\frac{\delta}{ds}\left(\frac{dx^\alpha}{ds}\right) \equiv \frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\lambda\sigma}\frac{dx^\lambda}{ds}\frac{dx^\sigma}{ds} = 0, \tag{1}$$

where the Christoffel symbols $\Gamma^\alpha_{\lambda\sigma}$ are computed for the metric tensor $g_{\alpha\beta}(x)$, which determines the given gravitational field and the corresponding Riemannian space V_4 with metric signature of the type $(---+)$.

Let us pose the question: does there exist another Riemannian space $\bar{V}_4$, in which, in a common coordinate system, the curves (1) are integral curves of equations of motion of the form

$$\frac{\delta}{d\tau}\left(\frac{dx^\alpha}{d\tau}\right) \equiv \frac{d^2x^\alpha}{d\tau^2} + \bar{\Gamma}^\alpha_{\lambda\sigma}\frac{dx^\lambda}{d\tau}\frac{dx^\sigma}{d\tau} = F^\alpha\left(x, \frac{dx^\sigma}{d\tau}\right), \tag{2}$$

where $\bar{\Gamma}^\alpha_{\lambda\sigma}$ are the Christoffel symbols corresponding to the metric $\bar{g}_{\alpha\beta}(x)$ in $\bar{V}_4$? If such a $\bar{V}_4$ is found for a given structure of the gravitational force F^α, then we shall say that the field V_4 (the original) is modeled in $\bar{V}_4$ with preservation of the paths of test bodies.

If, from a formal mathematical point of view, it is of interest to seek as the model the most general form of a modeling manifold (spaces of affine connection, Finsler spaces, etc.), then for physics, apparently, the inverse formulation of the problem will be of greatest interest: the model should not be more complicated than the original. Therefore it has been assumed above that the model is $\bar{V}_4$—a Riemannian space of the same signature as V_4, and, moreover, taking into account the physical interpretation, we require that (2) determine timelike or isotropic curves in $\bar{V}_4$. In this connection, two approaches to the problem are possible: 1) **all** timelike and isotropic geodesics of V_4 are modeled (complete modeling), and 2) only **some distinguished set** of timelike or isotropic geodesics of V_4 is modeled (selective, selection modeling). Each of these approaches deserves separate study, and both may prove useful for different problems. In the present note only complete modeling is considered. Separate special cases of modeling of one kind or another have been considered in works ([4–10]).

1. Since in (2) the right-hand side contains the "force" F^α, it is first of all necessary to determine what restrictions are imposed on this vector under complete modeling. Since for the canonical parameter $\bar{\tau}$

$$\bar{g}_{\alpha\beta}\frac{dx^\alpha}{d\bar{\tau}}\frac{dx^\beta}{d\bar{\tau}} = \varepsilon, \tag{3}$$

where $\varepsilon = 1$ for timelike curves and $\bar{\tau} = \bar{s}$, and $\varepsilon = 0$ for isotropic ones, then, differentiating this equality covariantly along the path, we obtain

$$\bar{g}_{\alpha\beta}\frac{dx^\alpha}{d\bar{\tau}}\frac{\delta}{d\bar{\tau}}\left(\frac{dx^\beta}{d\bar{\tau}}\right) = \bar{g}_{\alpha\beta}\frac{dx^\alpha}{d\bar{\tau}}F^\beta = 0. \tag{4}$$

We shall show below that F^α can always be represented as a polynomial in $dx^\alpha/d\bar{\tau}$ of degree not higher than the 4th, i.e.

$$F^\alpha = v^\alpha + v^\alpha_\sigma u^\sigma + v^\alpha_{\lambda\sigma}u^\lambda u^\sigma + v^\alpha_{\lambda\sigma\tau}u^\lambda u^\sigma u^\tau + v^\alpha_{\lambda\sigma\tau\omega}u^\lambda u^\sigma u^\tau u^\omega, \tag{5}$$

where $u^\alpha \equiv dx^\alpha/d\bar{\tau}$, and $v^\alpha, \dots, v^\alpha_{\lambda\sigma\tau\omega}$ depend only on the point, but not on the 4-velocity u^α. Assuming that F^α is a polynomial of degree $k \leq 4$ in u^α and taking (3), (4) into account, for any k one can establish the necessary restrictions on the force. We shall need below only the cases $k = 0, 1, 2$, for which the following assertions hold:

A. Let u^α be a timelike vector ($\varepsilon = 1$). Then:

if $F^\alpha = v^\alpha$,

$$\text{then } v^\alpha = 0,$$

if $F^\alpha = v^\alpha + v^\alpha_\lambda u^\lambda$,

$$\text{then } v^\alpha = 0, \quad v_{(\alpha\beta)} = 0, \tag{6}$$

if $F^\alpha = v^\alpha + v^\alpha_\lambda u^\lambda + v^\alpha_{\lambda\sigma} u^\lambda u^\sigma$,

$$\text{then } v^\alpha = 0, \quad v_{(\alpha\beta)} = 0, \quad v_{(\alpha\beta\gamma)} = 0.$$

B. Let u^α be an isotropic vector ($\varepsilon = 0$). Then:

if $F^\alpha = v^\alpha$,

$$\text{then } v^\alpha = 0,$$

if $F^\alpha = v^\alpha + v^\alpha_\lambda u^\lambda$,

$$\text{then } v^\alpha = 0, \quad v_{\alpha\beta} = \bar{\sigma} g_{\alpha\beta} + \tilde{v}_{\alpha,\beta}, \quad \tilde{v}_{(\alpha\beta)} = 0,$$

if $F^\alpha = v^\alpha + v^\alpha_\lambda u^\lambda + v^\alpha_{\lambda\sigma} u^\lambda u^\sigma$,

$$\text{then } v^\alpha = 0, \quad v_{\alpha\beta} = \bar{\sigma} g_{\alpha\beta} + \tilde{v}_{\alpha,\beta}, \quad \tilde{v}_{(\alpha\beta)} = 0,$$

$$v_{(\alpha\beta\gamma)} = 0. \tag{7}$$

Analogous formulas can be obtained for $k = 3, 4$. All of them may be proved, for example, in the following way. Pass at some point $\bar{V}_4$ to a local orthonormal frame, relative to which $\bar{g}_{\alpha\alpha} = e_\alpha = \pm 1$, $\bar{g}_{\alpha\beta} = 0$ ($\alpha \neq \beta$); as a consequence of (3),

$$u^\alpha = \frac{dx^\alpha}{d\bar{\tau}} = \left(a_1, a_2, a_3, \pm \left(\varepsilon + \sum_i a_i^2 \right)^{1/2} \right),$$

where the a_i are completely arbitrary numbers, while the plus or minus sign before the root denotes the choice of direction on the modeling curve. If $\varepsilon = 1$, then, putting $\sum_i a_i^2 = \sigma < 1$, one may write

$$\left(\varepsilon + \sum_i a_i^2 \right)^{1/2} = 1 + \frac{\sigma}{2} - \frac{\sigma^2}{8} + \frac{\sigma^3}{16} - \ldots \tag{8}$$

Writing (4) and taking in (8) an expansion of sufficiently high order, collecting like terms and taking into account that the a_i (apart from the condition $\sigma < 1$) are arbitrary, we obtain that the coefficients of the different powers of a^i are equal to zero. These relations will have tensor form, equivalent to (6). For $\varepsilon = 0$ one may take $a_1 = 1$ and $a_2^2 + a_3^2 < 1$ and apply the same procedure.

2. Concerning the modeling paths, one may, in some domain of existence and uniqueness of the solution of system (2), put forward the following assumptions:

I. **At each point of V_4 an isotropic direction passes into an isotropic direction of $\overline{V}_4$, i.e. the light cone is preserved.** In this case one may use an almost obvious theorem: if at each point all isotropic directions of V_4 of signature $(---+)$ are isotropic directions also for $\overline{V}_4$, then their metrics are conformal. The assertion can, for example, be confirmed by using the method indicated above for the equations $g_{\alpha\beta}u^\alpha u^\beta = \overline{g}_{\alpha\beta}u^\alpha u^\beta = 0$, i.e. this assumption leads to a very narrow type of modeling—this is a conformal mapping of V_4 onto $\overline{V}_4$.

II. **All isotropic and timelike geodesics of V_4 are modeled by timelike curves (2) in $\overline{V}_4$.** This admits a physical interpretation: particles of photon type (more precisely, light lines), moving along geodesics in V_4 (the analogue of Newton' s first law—the law of inertia), in the presence of a "force" in $\overline{V}_4$ move nonfreely, not with maximal velocity (the analogue of the second law of dynamics).

III. **Only part of the isotropic geodesics of $\overline{V}_4$ is modeled by isotropic curves (2), while other isotropic geodesics of V_4 are modeled by timelike curves.** Such formally possible modeling leads to a difficult problem—the physical interpretation of such a situation. Here it would be necessary to explain the presence of two types of light lines (responding and not responding to the action of the exciting force).

Until an explanation of such a strange fact is obtained, modeling III hardly deserves analysis, which, however, can be carried out by a method analogous to that given below. In view of this we shall consider only modeling of type II. Then as a canonical parameter τ one may take the length of the arc $\bar{s} = \int d\bar{s} = \int (|\overline{g}_{\alpha\beta}dx^\alpha dx^\beta|)^{1/2}$. Here, in the sense $\tau = \tau(\bar{s})$ and $\tau' \equiv d\tau/d\bar{s} \neq 0$. Consequently,

$$\frac{dx^\alpha}{d\tau} = \frac{1}{\tau'}\frac{dx^\alpha}{d\bar{s}}, \qquad \frac{d^2x^\alpha}{d\tau^2} = \frac{1}{\tau u'^2}\frac{d^2x^\alpha}{d\bar{s}^2} - \frac{\tau''}{\tau u'^3}\frac{dx^\alpha}{d\bar{s}},$$

and equations (1) will be written in the form

$$\frac{d^2x^\alpha}{d\bar{s}^2} = \frac{\tau''}{\tau'}\frac{dx^\alpha}{d\bar{s}} - \Gamma^\alpha_{\lambda\sigma}\frac{dx^\lambda}{d\bar{s}}\frac{dx^\sigma}{d\bar{s}}$$

or

$$\frac{\delta}{d\bar{s}}\left(\frac{dx^\alpha}{d\bar{s}}\right) = \frac{\tau''}{\tau'}\frac{dx^\alpha}{d\bar{s}} + \Omega^\alpha_{\lambda\sigma}\frac{dx^\lambda}{d\bar{s}}\frac{dx^\sigma}{d\bar{s}}, \tag{9}$$

where $\Omega^\alpha_{\lambda\sigma} \equiv \overline{\Gamma}^\alpha_{\lambda\sigma} - \Gamma^\alpha_{\lambda\sigma}$ is the difference of two connection coefficients, i.e. a tensor. Contracting this equation with $\overline{g}_{\alpha\beta}\dfrac{dx^\beta}{d\bar{s}}$ and using (3), (4), where $\varepsilon = 1$, we obtain

$$\frac{\tau''}{\tau'} = -\Omega_{\lambda\sigma\tau}\frac{dx^\lambda}{d\bar{s}}\frac{dx^\sigma}{d\bar{s}}\frac{dx^\tau}{d\bar{s}},$$

i.e., substituting in (9), we obtain

$$\frac{\delta}{d\bar{s}}\left(\frac{dx^\alpha}{d\bar{s}}\right) = \Omega^\alpha_{\lambda\sigma}\frac{dx^\lambda}{d\bar{s}}\frac{dx^\sigma}{d\bar{s}} - \Omega_{\lambda\sigma\tau}\frac{dx^\lambda}{d\bar{s}}\frac{dx^\sigma}{d\bar{s}}\frac{dx^\tau}{d\bar{s}}\frac{dx^\alpha}{d\bar{s}}, \qquad \Omega_{\alpha\beta\gamma} \equiv \overline{g}_{\alpha\sigma}\Omega^\sigma_{\beta\gamma}. \tag{10}$$

Thus: **all timelike and isotropic geodesics of any gravitational field** (of any V_4 with signature of type $(- - - +)$) **can be modeled by timelike curves in any** $\overline{V}_4$ (of the same signature) **with equations of motion (10); in particular, it is very important to emphasize that as** $\overline{V}_4$ (the modeling space) **one can always take flat space; such modeling is always feasible and will be preferable in the sense of physical interpretations, since under such modeling the gravitational field is entirely "pumped over"

into a "gravitational force." From (10) there follows directly the assertion stated above that F^α is a polynomial in $dx^\alpha/d\bar{s}$ of degree not higher than the 4th. Condition (4) for (10) is satisfied identically.

3. One may pose the question: is it possible, for certain gravitational fields, to simplify the "force" standing on the right-hand side of (10)? More precisely: for which gravitational fields will the right-hand side of (10) be a polynomial of degree k, where $k < 4$? The complete answer to this question can be formulated as a theorem:

Theorem. The "force" standing on the right in (10) may be either of the 4th degree with respect to $dx^\alpha/d\bar{s}$, and then no conditions are imposed on V_4 and $\bar{V}_4$, or of the 2nd degree, and for this it is necessary and sufficient that

$$\Omega_{(\alpha\beta\gamma)} = W_{(\alpha}g_{\beta\gamma)}, \qquad W_\alpha \overset{\text{def}}{=} \frac{1}{6}\left(\Omega^\sigma_{\alpha\sigma} + 2\Omega^\sigma_{\sigma\alpha}\right), \quad \text{and then} \quad F^\alpha = v^\alpha_{\lambda\sigma}\frac{dx^\lambda}{d\bar{s}}\frac{dx^\sigma}{d\bar{s}},$$

where

$$v^\alpha_{\lambda\sigma} = \Omega^\alpha_{\lambda\sigma} - \frac{1}{2}\left(\delta^\alpha_\lambda W_\sigma + \delta^\alpha_\sigma W_\lambda\right).$$

The proof is based on applying the method indicated above and using (6); or else $F^\alpha \equiv 0$, and then we arrive at a geodesic mapping of gravitational fields—a problem solved completely in closed form in (11). Note that the force contains no terms of odd degree in $dx^\alpha/d\bar{s}$, i.e., for complete modeling a Lorentz-type force in electrodynamics is impossible. For selective modeling such a force is possible (9), and therefore this type of modeling deserves special study. As an example of complete modeling with a force of the 2nd degree, one may point to the case of conformal V_4 and $\bar{V}_4$.

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Note: Figure translations are in progress. See original paper for figures.