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ELLIPTIC  
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**Abstract**

**Full Text**

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*MATHEMATICS*

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## ON THE DISTRIBUTION OF THE EIGENVALUES OF THE OPERATOR OF A SELF-ADJOINT ELLIPTIC BOUNDARY-VALUE PROBLEM IN AN UNBOUNDED DOMAIN

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Asymptotic formulas characterizing the distribution of the eigenvalues of self-adjoint elliptic operators defined in all of space were obtained in papers (<sup>1-3</sup>). For operators generated by general elliptic boundary-value problems, the question of the distribution of eigenvalues has been studied by many authors (see (<sup>4-6</sup>), etc.). In these works boundary-value problems were considered in bounded domains. The case of an unbounded domain is treated in the note (<sup>7</sup>), in which the eigenvalues of operators of boundary-value problems for the Schrödinger equation are studied.

In the present paper we investigate the asymptotic distribution of the eigenvalues of a self-adjoint operator generated by a general elliptic boundary-value problem in an unbounded domain. Here, as in (<sup>1-4</sup>), a method is used that is based on employing certain properties of the Green's function of the corresponding parabolic boundary-value problem. In connection with this, we first obtain an estimate of the Green's matrix-function for a homogeneous boundary-value problem for a certain class of parabolic systems with coefficients growing at infinity.

In the constructions and proofs, substantial use is made of the methods of the works of A. G. Kostyuchenko and Yu. M. Sudarev, as well as the results of S. D. Eidelman and S. D. Ivasyshen (<sup>8,9</sup>).

1. Let  $\Omega$  be an infinite domain in the space  $E_n$  with finite or infinite boundary  $S$ . In the cylindrical domain  $Q = \Omega \times [0, T]$  consider the problem of finding a solution  $u(t, x) = \{u_1(t, x), \dots, u_N(t, x)\}$  of the strongly parabolic system

$$\frac{\partial u}{\partial t} + (-1)^b \sum_{|k|=2b} A_k(x) D_x^k u + \sum_{|k| \leq 2b-1} P_k(x) D_x^k u + Q(x) u \equiv$$

$$\equiv \frac{\partial u}{\partial t} + L_0(x, D_x)u + L_1(x, D_x)u + Q(x)u = 0, \quad (1)$$

satisfying the conditions

$$u|_{t=0} = f(x), \quad (2)$$

$$B_\nu(x, D_x)u|_\Gamma \equiv \sum_{j=1}^N \sum_{|k| \leq r_\nu} b_{\nu j}^{(k)}(x) D_x^k u_j|_\Gamma = 0, \quad (3)$$

where  $D_x^{kb} = \partial^{|k|} / \partial x_1^{k_1} \dots \partial x_n^{k_n}$ ,  $\Gamma = [0, T] \times S$ ,  $r_\nu \leq 2b - 1$ ,  $\nu = 1, 2, \dots, bN$ .

With respect to the coefficients of system (1) we introduce a number of assumptions, coinciding with the assumptions made in (2) for the case  $\Omega = E_n$ . Namely, we shall suppose that for all  $x \in \Omega$  the matrices  $A_k(x)$  and  $Q(x)$  are symmetric and the following inequalities hold:

$$|L_0(x, \xi) - L_0(y, \xi)| \leq c|x - y|^\gamma |\xi|^{2b}, \quad |x - y| \leq 1, \quad 0 < \gamma < 1; \quad (4)$$

$$|Q(x) - Q(y)| \leq C\lambda_1^\alpha(y)|x - y|, \quad |x - y| \leq 1, \quad \alpha < 1 + 1/2b; \quad (5)$$

$$|P_k(x)| \leq \lambda_1^{(2b-|k|)/2b-\varepsilon_0}(x), \quad \varepsilon_0 > 0; \quad (6)$$

$$\lambda_1(x) \leq C\lambda_1(y), \quad |x - y| \leq 1; \quad (7)$$

$$\lambda_n(x) \leq \lambda_1^\beta(x), \quad \beta \leq 1; \quad (8)$$

$$\lambda_n(x) \leq C \exp\{c|x - y|\lambda_1^{1/2b}(y)\}, \quad |x - y| > 1, \quad (9)$$

where  $\{\lambda_i(x)\}_{i=1}^N$  are the characteristic roots of the matrix  $Q(x)$ , numbered in increasing order.

The assumptions on the smoothness of the boundary  $S$  and of the coefficients  $b_{\nu j}^{(k)}(x)$  are the same as in (8); we also assume that the algebraic Lopatinski condition is satisfied.

**Theorem 1.** *If the conditions formulated above and inequalities (4)–(9) are fulfilled, then there exists a Green matrix  $G(t, x, \xi)$  of problem (1), (2), (3), for which the representation*

$$G(t, x, \xi) = Z_0(t, x - \xi, \xi) + V_1(t, x, \xi) + V_2(t, x, \xi), \quad (10)$$

holds, where  $Z_0(t, x - \xi, \eta)$  is the fundamental solution matrix of the system

$$du/dt + A_0(\eta, D_x)u + Q(\eta)u = 0 \quad (11)$$

with the coefficients “frozen” at the point  $\eta \in \Omega$ , and the matrices  $V_1(t, x, \xi)$  and  $V_2(t, x, \xi)$  satisfy the following inequalities:

$$|V_1(t, x, \xi)| \leq Ct^{-n/2b} \exp \left\{ -c \frac{[|x - \xi| + \rho(\xi, S)]^q}{t^{1/(2b-1)}} - t\lambda_1(\xi) \right\}, \quad (12)$$

$$|V_2(t, x, \xi)| \leq Ct^{-(n-\varepsilon)/2b} \exp \left\{ -c \frac{[|x - \xi| + \rho(\xi, S)]^q}{t^{1/(2b-1)}} - ct\lambda_1(\xi) \right\} \\ + C\lambda_1^{-l}(\xi) \exp \left\{ -c \frac{|x - \xi|^q}{t^{1/(2b-1)}} \right\}, \quad (13)$$

$\rho(\xi, S)$  is the distance from the point  $\xi$  to the boundary  $S$ , and  $l > 0$  is an arbitrarily large number.

The proof of the theorem is carried out by the “parametrix” method according to the scheme of <sup>(1)</sup>. We seek the matrix  $G(t, x, \xi)$  in the form

$$G(t, x, \xi) = G_0(t, x, \xi, \xi) + \int_0^t d\tau \int_{\Omega} G_0(t - \tau, x, y, y) \varphi(\tau, y, \xi) dy, \quad (14)$$

where  $G_0(t, x, \xi, \eta)$  is the Green matrix of the boundary-value problem (11), (2), (3). We choose the matrix  $\varphi(t, x, \xi)$  so that  $G(t, x, \xi)$ , as a function of  $t$  and  $x$ , is a solution of system (1). Then, by virtue of the homogeneity of the boundary conditions (3),  $G(t, x, \xi)$  will be the Green matrix of problem (1), (2), (3).

On the basis of the results of <sup>(8)</sup>, it is easy to show that for the matrix  $G_0(t, x, \xi, \xi)$  the following representation and estimates are valid:

$$G_0(t, x, \xi, \xi) = Z_0(t, x - \xi, \xi) + V_1(t, x, \xi), \quad (15)$$

$$|D_x^k G_0(t, x, \xi, \xi)| \leq Ct^{-(n+|k|)/2b} \exp \left\{ -c \frac{|x - \xi|^q}{t^{1/(2b-1)}} - t\lambda_1(\xi) \right\}, \quad (16)$$

where, for  $V_1(t, x, \xi)$ , (12) is fulfilled,  $|k| \leq 2b$ .

By virtue of these inequalities and the assumptions made, the solution of the integral equation for  $\varphi(t, x, \xi)$  and estimate (13) are obtained by the method set forth in <sup>(9)</sup>, p. 74, and <sup>(1)</sup>.

**Remark.** If one requires the existence of such an  $l > 0$  that

$$\int_{\Omega} \lambda_1^{-l}(x) dx < \infty, \quad (17)$$

then, on the basis of estimates (12), (13), (16), one may assert that the matrix  $G(t, x, \xi)$  is the kernel of a Hilbert-Schmidt operator.

II. Consider in the domain  $\Omega$  the boundary-value problem

$$Lu(x) = \varphi(x), \quad (18)$$

$$B_{\nu}u(x) = 0, \quad (19)$$

where  $L = L_0(x, D_x) + L_1(x, D_x) + Q(x)$  is a strongly elliptic operator of the left-hand side of system (1),  $u(x) = \{u_1(x), \dots, u_N(x)\}$ ,  $\nu = 1, 2, \dots, bN$ .

We assume that the boundary  $S$  and the coefficients of the operators  $L$  and  $B_{\nu}$  satisfy the conditions introduced in I. We shall now make additional assumptions ensuring the existence of a positive self-adjoint operator naturally associated with the boundary-value problem (18), (19). We require that the expression  $L$  be formally symmetric on finite functions in  $\Omega$ , and that the system of boundary operators  $B_{\nu}$  be normal and equivalent to its adjoint (see <sup>(10,11)</sup>). We also assume the positivity of the operator  $A$ , defined as follows:

$$D(A) = \{u, u \in C_0^{2b}(\Omega), B_{\nu}u|_S = 0, \nu = 1, 2, \dots, bN\}, \quad Au = Lu, \quad (20)$$

where  $C_0^{2b}(\Omega)$  is the set of vector-functions,  $2b$  times continuously differentiable in  $\Omega + S$  and finite at infinity.

In view of the imposed conditions, the operator  $A$  is symmetric, and its closure admits a self-adjoint extension in the Hilbert space  $\mathcal{L}_2(\Omega)$ .

The solution of the parabolic boundary-value problem can be written in the form

$$u(t, x) = e^{-tA}f(x), \quad (21)$$

whence it follows (see Remark 1) that the spectrum of the operator  $A$  is discrete and its eigenvalues increase as the number increases.

Our task is to find the asymptotics as  $\lambda \rightarrow +\infty$  of the function  $N(\lambda)$ —the number of eigenvalues of the operator  $A$  not exceeding  $\lambda$ . To this end we shall study the behavior of the trace of the matrix  $G(t, x, y)$  for small  $t$  and use the formula

$$\operatorname{tr} \int_{\Omega} G(t, x, x) dx = \int_0^{\infty} e^{-\lambda t} dN(\lambda). \quad (22)$$

The additional restrictions under which the required estimate can be obtained are imposed by the following

**Lemma.** If the function  $\lambda_1(x)$  in the domain  $\Omega$  satisfies the inequality

$$C_1(1 + |x|^\alpha) \leq \lambda_1(x) \leq C_2(1 + |x|^\alpha), \quad (23)$$

where  $\alpha > 0$  is arbitrary,  $C_2 > C_1 > 0$ , and there exists such a  $\delta > 0$  that for  $0 \leq l < \delta$  and any  $R > R_0 = \rho(0, S)$

$$\operatorname{mes}\{x, \rho(x, S) = l, R \leq |x| \leq R + 1\} \leq C_0 R^{n-1+\alpha/2b-\varepsilon} \quad (24)$$

( $C_0 > 0$ ,  $\varepsilon > 0$  do not depend on  $R$  and  $l$ ),

$$\int_{\Omega} \exp\left\{-\frac{\rho^a(x, S)}{t^{1/(2b-1)}} - t\lambda_1(x)\right\} dx \leq C t^{\varepsilon_1} \int_{\Omega} \exp\{-t\lambda_1(x)\} dx, \quad 0 < \varepsilon_1 \leq \varepsilon/\alpha. \quad (25)$$

Let us note that if the boundary of the unbounded domain  $\Omega$  is compact, then inequality (25) is valid even without the additional conditions (23), (24).

Using now Theorem 1, inequality (25), and the results of (3), it is not difficult to show that as  $t \rightarrow 0$

$$\operatorname{tr} \int_{\Omega} G(x, x, t) dx \sim \frac{1}{(2\pi)^n} \sum_{i=1}^N \int_{\Omega} dx \int_{E_n} \exp\{-\mu_i(\xi, x)t\} d\xi, \quad (26)$$

where  $\mu_i(\xi, x)$  are the eigenvalues of the matrix  $L_0(x, \xi) + Q(x)$ .

Following (3), let us write the following equality:

$$\int_{E_n} \exp\{-t\mu_i(x, \xi)\} d\xi = \int_{\lambda_i^*(x)}^{\infty} \exp\{-\lambda t\} d_{\lambda} \omega_i(x, \lambda), \quad (27)$$

where the functions  $(2\pi)^n \omega_i(x, \lambda)$ ,  $i = 1, 2, \dots, N$ , for each  $x \in \Omega$  are equal to the volume of the domain bounded by the corresponding surface  $\mu_i(\xi, x) = \lambda$  in the space of  $n$ -dimensional vectors  $\xi$ .

Imposing on the function

$$\Phi(\lambda) = \sum_{i=1}^N \int_{\{x, x \in \Omega, \lambda_i^*(x) < \lambda\}} \omega_i(x, \lambda) dx$$

the Tauberian condition of Korenblyum (see (3)), we arrive at the following result:

$$N(\lambda) \sim \sum_{i=1}^N \int_{\{x, x \in \Omega, \lambda_i^*(x) < \lambda\}} \omega_i(x, \lambda) dx \quad \text{as } \lambda \rightarrow +\infty. \quad (28)$$

In the case of a boundary-value problem for a single differential equation, formula (28) takes the more explicit form:

$$N(\lambda) \sim \frac{1}{(2\pi)^n \Gamma(n/2b + 1)} \int_{\{x, x \in \Omega, Q(x) < \lambda\}} F(x) (\lambda - Q(x))^{n/2b} dx, \quad (29)$$

where

$$F(x) = \int_{E_n} \exp\{-L_0(x, \xi)\} d\xi.$$

From all that has been set forth, the main result of the work follows.

**Theorem 2.** Let  $A$  be a positive self-adjoint operator generated by the elliptic boundary-value problem (18), (19) in an unbounded domain  $\Omega$  with infinite boundary  $S$ . If: a) the coefficients of the system (18) satisfy conditions (4)–(9), (17), (23), and the coefficients of the boundary operators (19) are sufficiently smooth; b) the boundary  $S$ , in addition to the usual smoothness conditions, satisfies condition (24); c) for the corresponding parabolic boundary-value problem (1), (2), (3) the Lopatinskii condition is fulfilled, then for the function  $N(\lambda)$  the asymptotic formula (28) holds.

If the boundary  $S$  is finite, then for formula (28) to be valid one need not require, in the stated conditions, satisfaction of inequalities (23), (24).

In the case of a single differential equation, under the assumptions made above, formula (29) is valid.

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*Note: Figure translations are in progress. See original paper for figures.*

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