ON A SPECIAL CLASS OF FINITE-DIMENSIONAL SUBSPACES OF A BANACH SPACE

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Abstract

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MATHEMATICS

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ON A SPECIAL CLASS OF FINITE-DIMENSIONAL SUBSPACES OF A BANACH SPACE

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Let E be a real Banach space and E_0 some subspace of it. Suppose that from the conditions

$$\|Ax\| \leq \|x\| \quad (x \in E), \qquad Ax \equiv x \quad (x \in E_0), \tag{1}$$

where A is a linear operator, there follows the identity

$$Ax \equiv x \quad (x \in E). \tag{2}$$

Then we shall say that the subspace E_0 has the e-property. A number of theorems on subspaces possessing the e-property were established in papers ([1,2]). Of main interest to us is the fact that in many infinite-dimensional spaces there are finite-dimensional subspaces possessing the e-property. Such subspaces arise naturally in a number of problems in approximation theory.

Of interest is the problem of as complete as possible a description of subspaces possessing the e-property in various concrete Banach spaces, and the problem of computing or estimating the minimal dimension of such subspaces. In the present paper these problems are solved for the spaces l_p^n, l_p, and L_p.

1. Let S and S^* be the unit spheres respectively in the Banach space E and in its conjugate E^*. One says that a functional $f \in S^*$ passes through a point $x \in S$ if $f(x) = 1$. A point $x_0 \in S$ is called a point of smoothness of S if through it there passes a unique functional $f_0 \in S^*$. One says that a subspace E_0 of the space H is saturated with points of smoothness of S if the set $F(E_0)$ of functionals passing through points of smoothness of S lying in E_0 is total. Such subspaces are called saturated (see ([1,2])).

A saturated subspace has the e-property (cf. ([1,2])). Indeed, let A be an extension of the identity operator from E_0 to E without increasing the norm ($\|A\| = 1$). For any functional $f_0 \in F(E_0)$ there is a point $x_0 \in E_0$ such that $f_0(x_0) = 1$. Consider the functional $g_0(x) = f_0(Ax)$ $(x \in E)$. Since $|g_0(x)| = |f_0(Ax)| \leq$

$\|Ax\| \leq \|x\|$ and $g_0(x_0) = f_0(Ax_0) = f_0(x_0) = 1$, it follows that $g_0 \in S^*$ and passes through the point of smoothness x_0. Consequently, $g_0 \equiv f_0$, and $f_0(Ax - x) = 0$ for $x \in E$. The totality of the set $F(E_0)$ entails $Ax \equiv x$ $(x \in E)$.

2. A description of saturated subspaces in finite-dimensional spaces with metric l_p $(1 \leq p < \infty)$ is given by the following theorems.

Theorem 1. Let p not be an even integer. Then, in order that in the space l_p^n the linearly independent vectors

$$e_1 = \{\xi_{11}, \xi_{12}, \dots, \xi_{1n}\}; \dots; e_k = \{\xi_{k1}, \xi_{k2}, \dots, \xi_{kn}\} \tag{3}$$

form a basis of a saturated subspace, it is necessary and sufficient that the k-dimensional vectors

$$g_1 = \{\xi_{11}, \xi_{21}, \dots, \xi_{k1}\}; \dots; g_n = \{\xi_{1n}, \xi_{2n}, \dots, \xi_{kn}\} \tag{4}$$

be pairwise non-collinear.

Thus, for example, the subspace spanned by the vectors

$$e_1 = \{0, 1, \dots, n(n-1)/2\}; \qquad e_2 = \{0, 1, \dots, (n-1)^2\};$$

$$e_3 = \{1, 3, \dots, 3^{n-1}\}$$

of the space l_p^n $(n \geq 4,\ 1 \leq p < \infty,\ p$ is not an even integer) is ⋯

saturated subspace, while the subspace spanned by any two of the indicated vectors is not a saturated subspace.

It follows from Theorem 1 that the vectors

$$e_1 = \{1, 2, \dots, n\}, \qquad e_2 = \{1, 2^2, \dots, n^2\} \tag{5}$$

form a basis of a two-dimensional saturated subspace in l_p^n, if p is not an even integer.

Let now p be an even integer. In this case the space l_p^n may have no proper saturated subspaces. For example, the space l_2^n is Hilbert, and therefore it has no saturated subspaces. Below, by φ_j $(j = 1, \dots, C_{p+k-2}^{k-1})$ we denote all distinct homogeneous polynomials

$$\varphi_j(x_1, \dots, x_k) = x_1^{l_1} \cdots x_k^{l_k} \left(0 \leq l_m \leq p-1,\ \sum_{m=1}^{k} l_m = p-1\right) \tag{6}$$

of degree $p-1$ in k variables $x_1, \dots, x_k$.

Theorem 2. *Let p be an even integer. In order that k-dimensional saturated subspaces exist in the space l_p^n, it is necessary and sufficient that the condition*

$$C_{p+k-2}^{k-1} \geq n. \tag{7}$$

be satisfied.

From this theorem, in particular, it follows that in l_p^n (p an even integer) two-dimensional saturated subspaces exist if and only if $p \geq n$.

Theorem 3. *Let p be an even integer. Then, in order that in the space l_p^n linearly independent vectors*

$$e_1 = \{\xi_{11}, \xi_{12}, \dots, \xi_{1n}\}, \dots, e_k = \{\xi_{k1}, \xi_{k2}, \dots, \xi_{kn}\} \tag{8}$$

form a basis of a saturated subspace, it is necessary and sufficient that the rank of the matrix with elements

$$a_{ij} = \varphi_j(\xi_{1i}, \xi_{2i}, \dots, \xi_{ki}) \qquad (i = 1, \dots, n;\ j = 1, \dots, C_{p+k-2}^{k-1}) \tag{9}$$

be equal to n.

For example, if $n \leq 4$, then the vectors (5) form a basis of a saturated subspace in l_4^n.

We give one more assertion.

Theorem 4. *If in l_p^n ($1 \leq p < \infty$) there exist two-dimensional saturated subspaces, then every saturated subspace contains a two-dimensional saturated subspace.*

3. We pass to the consideration of the space l_p.

Theorem 5. *Let p not be an even integer, $1 \leq p < \infty$. Let*

$$e_1 = \{\xi_1, \xi_2, \dots\}, \qquad e_2 = \{\eta_1, \eta_2, \dots\}. \tag{10}$$

Suppose that the sequence of two-dimensional vectors

$$\{\xi_i, \eta_i\} \qquad (i = 1, 2, \dots) \tag{11}$$

contains no pairwise collinear vectors and that every subsequence of the sequence of numbers

$$\alpha_i = \xi_i / \eta_i \qquad (i = 1, 2, \dots) \tag{12}$$

of the extended number line contains isolated points (of this same subsequence).

Then the vectors (10) form a basis of a two-dimensional saturated subspace in the space l_p.

Thus, for example, the subspace spanned by the vectors

$$e_1 = \left\{\frac{1}{2}, \dots, \frac{1}{2^n}, \dots\right\}, \qquad e_1 = \left\{\frac{1}{3}, \dots, \frac{1}{3^n}, \dots\right\}$$

is a saturated subspace of l_p $(1 \leq p < \infty)$, if p is not an even integer. Thus, in the space l_p for p, not

which is an even integer, there always exists a two-dimensional saturated subspace.

In the case where p is an even integer, it follows from the results obtained in (3) that in l_p there are no finite-dimensional saturated subspaces. However, the following is true.

Theorem 6. *Let $p \neq 2$. Then in l_p there are saturated subspaces of any finite defect.*

4. The following theorem establishes the existence of two-dimensional saturated subspaces in some spaces L_p $(1 \leq p < \infty)$.

Theorem 7. *Let p be an odd integer. Let $u(t), v(t) \in L_p[0,1]$. Let the function $\varphi(t) = u(t)/v(t)$ be defined and finite almost for all $t \in [0,1]$, and suppose, moreover, that the following conditions are satisfied:*

1) for some set $D_0 \subseteq [0,1]$ of full measure the function $\varphi(t)$ establishes a one-to-one correspondence between the points of D_0 and $D^0 = \varphi(D_0)$.

2) there exists a finite or countable system of intervals $\{a_k, b_k\}$ such that D_0 is contained in their union and on each of the sets $\{a_k, b_k\} \cap D_0$ the function $\varphi(t)$ is strictly monotone;

3) there exists a finite or countable system of intervals $\{c_j, d_j\}$ such that D^0 is contained in their union and on each of the sets $\{c_j, d_j\} \cap D^0$ the function $\psi(s)$, inverse to the function $\varphi(t)$, is strictly monotone.

Then the functions $u(t)$ and $v(t)$ form a basis of a two-dimensional saturated subspace in the space $L_p[0,1]$.

We note that the conditions of Theorem 7 are satisfied for continuous $u(t)$ and $v(t)$ if, in particular, the function $\varphi(t)$ is strictly monotone. It follows from Theorem 7, for example, that the subspaces spanned by the vectors $u(t) \equiv 1$, $v(t) \equiv t$, or $u(t) \equiv \sin \pi t$, $v(t) \equiv \cos \pi t$, are saturated subspaces in $L_p[0,1]$, where p is an odd integer.

In (3) it was shown that in the case of an even integer p there are no saturated finite-dimensional subspaces in the spaces L_p.

5. We describe a broader class of subspaces possessing the e-property. We shall call a point $x \in S$ a **point of quasi-smoothness** if the functionals $f_x \in S^*$ passing through it form a compact set F_x. Let M_0 be the set of points of quasi-smoothness of S lying in the subspace E_0. Denote by $\mathfrak{F}(E_0)$ the collection of sets of functionals from S^*, each of which contains, for one functional each, the sets F_x $(x \in M_0)$.

Theorem 8. *Let each set in $\mathfrak{F}(E_0)$ be total. Then E_0 has the e-property.*

The question remains open of necessary and sufficient conditions (in terms of properties of the set S) for the subspace E_0 to have the e-property.

6. Following A. Lazar and M. Zippin (see (3)), denote by A_1 the class of Banach spaces E possessing the following property: for every finite-dimensional subspace $E_0 \subset E$ there exists a subspace $F \subset E^*$ of infinite defect such that

$$\|x\| = \sup_{f \in F \cap S^*} |f(x)| \qquad (x \in E_0). \tag{13}$$

Obviously, every space in A_1 contains no finite-dimensional saturated subspaces.

Theorem 9. *Let the Banach space E with smooth sphere S not belong to the class A_1. Then E contains finite-dimensional saturated subspaces.*

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Note: Figure translations are in progress. See original paper for figures.