ON THE PROBLEM OF CONTROL ON SUBSPACES IN INTERCONNECTED SYSTEMS

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ON THE PROBLEM OF CONTROL ON SUBSPACES IN INTERCONNECTED SYSTEMS

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In the present work we consider the problem of controlling interconnected systems in the presence of an additional requirement that, in the course of control, even before being brought to the prescribed final state, the system be brought to some subspace which it would not leave until the end of the control process.

Consider a controllable system described by the differential equations

$$\begin{aligned} \dot{x}_1 &= A_{11}x_1 + A_{12}x_2 + \dots + A_{1m}x_m, \\ &\dots \\ \dot{x}_m &= A_{m1}x_1 + A_{m2}x_2 + \dots + A_{mm}x_m + \mu(x_{m+1} - x_{2m+1}) + u(t), \\ \dot{x}_{m+1} &= A_{11}x_{m+1} + A_{12}x_{m+2} + \dots + A_{1m}x_{2m}, \\ &\dots \\ \dot{x}_{2m} &= A_{m1}x_{m+1} + A_{m2}x_{m+2} + \dots + A_{mm}x_{2m} + \mu(x_{2m+1} - x_1) + v(t), \\ \dot{x}_{2m+1} &= A_{11}x_{2m+1} + A_{12}x_{2m+2} + \dots + A_{1m}x_n, \\ &\dots \\ \dot{x}_n &= A_{m1}x_{2m+1} + A_{m2}x_{2m+2} + \dots + A_{mm}x_n + \mu(x_{m+1} - x_1) + w(t), \end{aligned} \quad (1)$$

where $n = 3m$. Here $x_1, \dots, x_n$ are the phase coordinates of the system; $u(t)$, $v(t)$, and $w(t)$ are controls applied to the system, which must satisfy the constraints

$$|u(t)| \le k; \qquad |v(t)| \le k; \qquad |w(t)| \le k. \quad (2)$$

If we introduce the matrices

$$A = \left\| \begin{matrix} A_{11} & A_{12} & \dots & A_{1m} \\ \cdot & \cdot & \cdot & \cdot \\ A_{m1} & A_{m2} & \dots & A_{mm} \end{matrix} \right\|, \qquad C = \left\| \begin{matrix} 0 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & 0 \\ \mu & 0 & \dots & 0 \end{matrix} \right\|,$$

$$x^* = \left\| \begin{matrix} x_1 \\ x_2 \\ \dots \\ x_m \end{matrix} \right\|, \quad x^{**} = \left\| \begin{matrix} x_{m+1} \\ x_{m+2} \\ \dots \\ x_{2m} \end{matrix} \right\|, \quad x^{***} = \left\| \begin{matrix} x_{2m+1} \\ x_{2m+2} \\ \dots \\ x_n \end{matrix} \right\|, \quad R = \left\| \begin{matrix} 0 \\ 0 \\ \dots \\ 0 \\ 1 \end{matrix} \right\|, \tag{3}$$

then equations (1) take the form

$$\begin{aligned} \dot{x}^* &= Ax^* + Cx^{**} - Cx^{***} + Ru(t), \\ \dot{x}^{**} &= Ax^{**} + Cx^{***} - Cx^* + Rv(t), \\ \dot{x}^{***} &= Ax^{***} + Cx^{**} - Cx^* + Rw(t). \end{aligned} \tag{4}$$

or

$$f(D)x = G\rho(t) \qquad (D = d/dt), \tag{5}$$

where

$$x = \left\| \begin{matrix} x_1 \\ \dots \\ x_n \end{matrix} \right\|, \quad f(D) = \left\| \begin{matrix} f_0(D) & -C & C \\ C & f_0(D) & -C \\ C & -C & f_0(D) \end{matrix} \right\|, \quad G = \left\| \begin{matrix} R & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & R \end{matrix} \right\|, \quad \rho(t) = \left\| \begin{matrix} u(t) \\ v(t) \\ w(t) \end{matrix} \right\|; \tag{6}$$

$$f_0(D) = ED - A, \tag{7}$$

and E is the identity matrix. As is seen from (6), $f(D)$ and G are block matrices. The determinants of the matrices $f_0(\lambda)$ and $f(\lambda)$ will be denoted by $\Delta_0(\lambda)$ and $\Delta(\lambda)$. By $F_0(\lambda) = \|F^0_{jk}(\lambda)\|$ and $F(\lambda) = \|F_{jk}(\lambda)\|$ we denote the adjugate matrices for $f_0(\lambda)$ and $f(\lambda)$, respectively. Transforming the matrix $f(\lambda)$ to triangular form, we find that $\Delta(\lambda) = \det\varphi(\lambda)\det\psi(\lambda)\Delta_0(\lambda)$, where $\varphi(\lambda) = f_0(\lambda) + C$, $\psi(\lambda) = f_0(\lambda) - C$. Since $\det\varphi(\lambda) = \Delta_0(\lambda) + \mu F^0_{1m}(\lambda)$, $\det\psi(\lambda) = \Delta_0(\lambda) - \mu F^0_{1m}(\lambda)$, the determinant of the matrix $f(\lambda)$ takes the form

$$\Delta(\lambda) = [\Delta_0(\lambda) + \mu F^0_{1m}(\lambda)]\,[\Delta_0(\lambda) - \mu F^0_{1m}(\lambda)]\,\Delta_0(\lambda). \tag{8}$$

Here $F^0_{1m}(\lambda)$ is the cofactor of the element $f^0_{m1}(\lambda)$ in the determinant of the matrix $f_0(\lambda)$. Denoting by λ_j $(j = 1, \dots, m)$ the zeros of the polynomial $\Delta_0(\lambda) + \mu F^0_{1m}(\lambda)$, by λ_{m+j} $(j = 1, \dots, m)$ the zeros of the polynomial $\Delta_0(\lambda) - \mu F^0_{1m}(\lambda)$, and by λ_{2m+j} $(j = 1, \dots, m)$ the zeros of the polynomial $\Delta_0(\lambda)$, we shall have the relation

$$\Delta_0(\lambda_s) = \begin{cases} -\mu F^0_{1m}(\lambda_s) & (s = 1, \dots, m), \\ \mu F^0_{1m}(\lambda_s) & (s = m+1, \dots, 2m), \\ 0 & (s = 2m+1, \dots, n). \end{cases} \tag{9}$$

Taking into account that $F(\lambda) = \Delta(\lambda) f^{-1}(\lambda)$, it can be shown that

$$F(\lambda) = \begin{Vmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{Vmatrix}, \tag{10}$$

where the block matrices σ_{jk} $(j, k = 1, 2, 3)$ have the form

$$\begin{gathered} \sigma_{11} = F_0(\lambda) \det\varphi \det\psi, \qquad \sigma_{21} = \sigma_{31} = -\Psi(\lambda) C F_0(\lambda) \det\varphi, \\ \sigma_{12} = \sigma_{32} = \Phi(\lambda) C F_0(\lambda) \det\psi, \qquad \sigma_{22} = F_0(\lambda) \det\varphi \det\psi, \\ \sigma_{13} = -\rho_1 + \rho_2 + \rho_3 - \rho_4, \qquad \sigma_{23} = \rho_1 + \rho_2 - \rho_3 - \rho_4, \\ \sigma_{33} = \rho_1 + \rho_2 + \rho_3 - \rho_4. \end{gathered} \tag{11}$$

Here $\rho_1 = \frac{1}{2}\Psi(\lambda)\Delta_0(\lambda)\det\varphi$, $\rho_2 = \frac{1}{2}F_0(\lambda)C\Psi(\lambda)\det\varphi$, $\rho_3 = \frac{1}{2}\Phi(\lambda)\Delta_0(\lambda)\det\psi$, $\rho_4 = \frac{1}{2}F_0(\lambda)C\Phi(\lambda)\det\psi$, and by $\Phi(\lambda)$ and $\Psi(\lambda)$ are denoted the adjugate matrices for the matrices $\varphi(\lambda)$ and $\psi(\lambda)$, respectively.

We shall further assume that all zeros of the polynomial $\Delta(\lambda)$ are simple, and denote by X_k the vectors that are formed from the n-th column of the matrix $F(\lambda_k) = [F(\lambda)]_{\lambda=\lambda_k}$ $(k = 1, \dots, n)$ by dividing all elements of this column by the element $F_{2m+1,n}(\lambda_k)$. Denoting

$$\frac{F^0_{jm}(\lambda_k)}{F^0_{1m}(\lambda_k)} = \begin{cases} p_{jk} \text{ for } j = 1, \dots, m; \ k = 1, \dots, m, \\ q_{jk} \text{ for } j = 1, \dots, m; \ k = m+1, \dots, 2m, \\ r_{jk} \text{ for } j = 1, \dots, m; \ k = 2m+1, \dots, n, \end{cases} \tag{12}$$

one can represent the vectors X_k in the form

$$X_k = \begin{cases} (p_{1k}, p_{2k}, \dots, p_{mk}, 0, 0, \dots, 0, p_{1k}, p_{2k}, \dots, p_{mk}) \\ \qquad (k = 1, \dots, m), \\ (0, 0, \dots, 0, q_{1k}, q_{2k}, \dots, q_{mk}, q_{1k}, q_{2k}, \dots, q_{mk}) \\ \qquad (k = m+1, \dots, 2m), \\ (r_{1k}, r_{2k}, \dots, r_{mk}, r_{1k}, r_{2k}, \dots, r_{mk}, r_{1k}, r_{2k}, \dots, r_{mk}) \\ \qquad (k = 2m+1, \dots, n). \end{cases} \tag{13}$$

The matrix X of type $n \times n$, whose columns are the vectors X_k $(k = 1, \dots, n)$, can be represented in the form of a block matrix

$$X = \begin{Vmatrix} p & 0 & r \\ 0 & q & r \\ p & q & r \end{Vmatrix}, \tag{14}$$

where $p = \|p_{jk}\|$, $q = \|q_{jk}\|$, $r = \|r_{jk}\|$ are matrices of type $m \times m$, whose elements p_{jk}, q_{jk}, r_{jk} are defined according to (12). The inverse matrix X^{-1} will be

$$X^{-1} = \begin{Vmatrix} 0 & -P & P \\ -Q & 0 & Q \\ R & R & -R \end{Vmatrix}, \tag{15}$$

where $P = p^{-1}$, $Q = q^{-1}$, $R = r^{-1}$. Thus, we have

$$X^{-1} = \begin{Vmatrix} 0 & 0 & \dots & 0 & -P_{11} & -P_{12} & \dots & -P_{1m} & P_{11} & P_{12} & \dots & P_{1m} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & -P_{m1} & -P_{m2} & \dots & -P_{mm} & P_{m1} & P_{m2} & \dots & P_{mm} \\ -Q_{11} & -Q_{12} & \dots & -Q_{1m} & 0 & 0 & \dots & 0 & Q_{11} & Q_{12} & \dots & Q_{1m} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ -Q_{m1} & -Q_{m2} & \dots & -Q_{mm} & 0 & 0 & \dots & 0 & Q_{m1} & Q_{m2} & \dots & Q_{mm} \\ R_{11} & R_{12} & \dots & R_{1m} & R_{11} & R_{12} & \dots & R_{1m} & -R_{11} & -R_{12} & \dots & -R_{1m} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ R_{m1} & R_{m2} & \dots & R_{mm} & R_{m1} & R_{m2} & \dots & R_{mm} & -R_{m1} & -R_{m2} & \dots & -R_{mm} \end{Vmatrix}. \tag{16}$$

Let B_k $(k = 1, \dots, n)$ denote the $(2m + 1)$-st row of the matrix $F(\lambda_k)$. In accordance with (10), we shall have

$$B_k = \begin{cases} \| \dots \quad 0 \quad \dots \quad -2\,[F^0_{1m}(\lambda_k)]^3\mu^2 \quad \dots \quad 2\,[F^0_{1m}(\lambda_k)]^3\mu^2 \, \|, & (k = 1, \dots, m), \\ \| \dots \quad -2\,[F^0_{1m}(\lambda_k)]^3\mu^2 \quad \dots \quad 0 \quad \dots \quad 2\,[F^0_{1m}(\lambda_k)]^3\mu^2 \, \|, & (k = m+1, \dots, 2m), \\ \| \dots \quad -2\,[F^0_{1m}(\lambda_k)]^3\mu^2 \quad \dots \quad -[F^0_{1m}(\lambda_k)]^3\mu^2 \quad \dots \quad [F^0_{1m}(\lambda_k)]^3\mu^2 \, \|, & (k = 2m+1, \dots, n), \end{cases} \tag{17}$$

where the elements located in the columns numbered m, $2m$, and $3m = n$ have been written explicitly. Since all zeros of the polynomial $\Delta(\lambda)$ are assumed to be simple, the vectors X_k and the row matrices B_k satisfy the relation

$$X_k B_k = F(\lambda_k) \qquad (k = 1, \dots, n). \tag{18}$$

We now introduce the variables $\xi_1, \dots, \xi_n$ by means of the linear transformation $x = X\xi$, where $\xi = (\xi_1, \dots, \xi_n)$. Hence, in accordance with (16), we shall have

$$\begin{aligned}
\xi_1 &= P_{11}(-x_{m+1} + x_{2m+1}) + P_{12}(-x_{m+2} + x_{2m+2}) + \dots \\
&\quad \dots + P_{1m}(-x_{2m} + x_n), \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
\xi_m &= P_{m1}(-x_{m+1} + x_{2m+1}) + P_{m2}(-x_{m+2} + x_{2m+2}) + \dots \\
&\quad \dots + P_{mm}(-x_{2m} + x_n), \\
\xi_{m+1} &= Q_{11}(-x_1 + x_{2m+1}) + Q_{12}(-x_2 + x_{2m+2}) + \dots \\
&\quad \dots + Q_{1m}(-x_m + x_n), \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
\xi_{2m} &= Q_{m1}(-x_1 + x_{2m+1}) + Q_{m2}(-x_2 + x_{2m+2}) + \dots \\
&\quad \dots + Q_{mm}(-x_m + x_n).
\end{aligned} \tag{19}$$

$$\begin{gathered}
\xi_{2m+1} = R_{11}(x_1 + x_{m+1} - x_{2m+1}) + R_{12}(x_2 + x_{m+2} - x_{2m+2}) + \dots \\
\dots + R_{1m}(x_m + x_{2m} - x_n),
\end{gathered} \tag{19a}$$

$$\begin{gathered}
\xi_n = R_{m1}(x_1 + x_{m+1} - x_{2m+1}) + R_{m2}(x_2 + x_{m+2} - x_{2m+2}) + \dots \\
\dots + R_{mm}(x_m + x_{2m} - x_n).
\end{gathered} \tag{19b}$$

The differential equations satisfied by the new variables $\xi_1, \dots, \xi_n$ will be

$$\frac{d\xi_j}{dt} = \lambda_j \xi_j + \frac{1}{\Delta_j(\lambda_j)} \left[B_{jm} u(t) + B_{j,2m} v(t) + B_{jn} w(t) \right] \quad (j = 1, \dots, n), \tag{20}$$

where B_{jm}, $B_{j,2m}$, and B_{jn} are the elements, located in columns with numbers m, $2m$, and $3m = n$, of the row matrices B_j. By $\Delta_j(\lambda_j)$ is denoted

$$\Delta_j(\lambda_j) = \left[\frac{\Delta(\lambda)}{\lambda - \lambda_j} \right]_{\lambda = \lambda_j}.$$

In accordance with (17), equations (20) take the form

$$\frac{d\xi_j}{dt} = \begin{cases} \lambda_j \xi_j + \nu_j[-v(t) + w(t)] & \text{for } j = 1, \dots, m, \\ \lambda_j \xi_j + \nu_j[-u(t) + w(t)] & \text{for } j = m+1, \dots, 2m, \\ \lambda_j \xi_j - \frac{1}{2}\nu_j[u(t) + v(t) - w(t)] & \text{for } j = 2m+1, \dots, n, \end{cases} \tag{21}$$

where

$$\nu_j = 2 \left[F^0_{1m}(\lambda_j) \right]^3 \mu^2 / \Delta_j(\lambda_j).$$

From what has been set out above there follows the following theorem on sub-optimal control (by which here is meant optimal control on subspaces) for interconnected systems, formulated and proved for brevity only for the time-optimal case.

Theorem. *System (1), from a given initial phase state $x(0)$, can be transferred suboptimally in time T to the origin of coordinates. On the half-interval $0 \leq t < t_1$, $t_1 \in (0, T)$, the controls are chosen in accordance with L. S. Pontryagin' s maximum principle* [1] *under the additional condition*

$$w(t) \equiv 0, \qquad 0 \leq t < t_1. \tag{22}$$

In this case the control $v(t)$, $0 \leq t < t^*$, is chosen according to the maximum principle so that, at the earliest possible time t^*, the state $\xi_1(t^*) = 0, \dots, \xi_m(t^*) = 0$ is realized. The control $u(t)$, $0 \leq t < t^{**}$, is chosen according to the maximum principle so that, at the earliest possible time t^{**}, the state $\xi_{m+1}(t^{**}) = 0, \dots, \xi_{2m}(t^{**}) = 0$ is realized. The larger of the numbers t^*, t^{**} is denoted here by t_1. If $t^* < t_1$, then on the half-interval $[t^*, t_1)$ we set $v(t) \equiv 0$. If, however, $t^{**} < t_1$, then on the half-interval $[t^{**}, t_1)$ we set $u(t) \equiv 0$. Under the control described above, system (1) is brought at the time t_1 to the manifold

$$\{x_1 = x_{m+1} = x_{2m+1}, \dots, x_m = x_{2m} = x_n\}. \tag{23}$$

On the interval $t_1 \leq t \leq T$, the controls are chosen in accordance with L. S. Pontryagin' s maximum principle under the additional condition

$$u(t) \equiv v(t) \equiv w(t). \tag{24}$$

In this case control is effected only along the coordinates $\xi_{2m+1}, \dots, \xi_n$. By choosing the control according to the maximum principle, the system is brought, under condition (24), in the minimal time $T - t_1$ to the origin of coordinates; moreover, on the time interval $t_1 \leq t \leq T$ the system does not leave the manifold (23).

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CITED LITERATURE

1. L. S. Pontryagin, V. G. Boltyanskii et al., *The Mathematical Theory of Optimal Processes*, Moscow, 1961.