SOLUTION OF DISCRETE PROGRAMMING PROBLEMS BY THE METHOD OF CONSTRUCTING A SEQUENCE OF PLANS

MATHEMATICS

1969

SovietRxiv

Abstract

Full Text

UDC 512.251.26+519.3:330.115

MATHEMATICS

V. A. EMELICHEV, V. I. KOMLIK

SOLUTION OF DISCRETE PROGRAMMING PROBLEMS BY THE METHOD OF CONSTRUCTING A SEQUENCE OF PLANS

(Presented by Academician L. V. Kantorovich, January 29, 1969)

Consider problem A of minimizing the functional

$$F(X) = g_1(x_1)\omega g_2(x_2)\omega \dots \omega g_m(x_m)\omega Q(x_1, x_2, \dots, x_m)$$

subject to the constraints

$$\sum_{i=1}^{m} a_i x_i \le c; \tag{1}$$

$$X = (x_1, x_2, \dots, x_m) \in G, \tag{2}$$

$$x_i \in \gamma_i = \{a_{i1}, a_{i2}, \dots, a_{is_i}\}, \quad i = 1, \dots, m, \tag{3}$$

where a_i, c, a_{ij} are positive integers; ω is the operation of addition or multiplication; $Q(X)$ is a function for which there exist functions $t_i(x_i)$, $i = 1, \dots, m$, such that

$$t_1(x_1)\omega t_2(x_2)\omega \dots \omega t_m(x_m) \le Q(X)$$

for any plan X of problem A. The set G may be specified in various ways (by equations, inequalities, logical conditions, etc.). Denote the set of plans of problem A by M, and the set of plans determined by constraints (1) and (3) by $\overline{M}$.

In the present note we propose a method for solving problem A, consisting in the construction and analysis of a sequence of plans of problem $\overline{\text{A}}$ for minimizing the functional $\overline{F}(X) = \overline{g}_1(x_1)\omega\overline{g}_2(x_2)\omega \dots \omega\overline{g}_m(x_m)$ on the set $\overline{M}$. Here $\overline{g}_i(x_i) = g_i(x_i)\omega t_i(x_i)$, $i = 1, \dots, m$.

Let $f_h(z)$ be the optimal value of the functional of problem $\overline{\text{A}}$, in which the index m and the parameter c are replaced respectively by h and z ([1]). A plan $X_0 = (x_1^0, x_2^0, \dots, x_m^0)$ of problem $\overline{\text{A}}$ will be called h-optimal ($1 \leq h \leq m$) and denoted by

$$(*, *, \dots, *, x_{h+1}^0, \dots, x_m^0),$$

if

$$\overline{F}(X_0) = \overline{g}_{h+1}(x_{h+1}^0)\omega \dots \omega \overline{g}_m(x_m^0)\omega f_h\left(c - \sum_{i=h+1}^{m} a_i x_i^0\right).$$

In this case the set of plans of problem $\overline{\text{A}}$ such that $x_i = x_i^0$, $i = h+1, \dots, m$, will be denoted by

$$\overline{M}(*, *, \dots, *, x_{h+1}^0, \dots, x_m^0) = \overline{M}(X_0).$$

Directly from the definition of h-optimality of the plan X_0 it follows that

Lemma 1. $\overline{F}(X_0) = \min\limits_{X \in \overline{M}(X_0)} \overline{F}(X)$.

We describe an algorithm φ for constructing a sequence of plans of problem A.

First step. Among the set W_1 of all $(m-1)$-optimal pla-

of problem $\bar{A}$ we find such a plan

$$X_1 = (*, *, \dots, *, x_m^1) = (x_1^1, x_2^1, \dots, x_m^1),$$

such that

$$\bar{F}(X_1) = \min_{X \in W_1} \bar{F}(X).$$

k-th step ($k = 2, 3, \dots$). We transform the set W_{k-1} into the set W_k according to the following rule. We leave unchanged all plans of the set W_{k-1}, except for the plan

$$X_{k-1} = (*, *, \dots, *, x_p^{k-1}, \dots, x_m^{k-1}) = (x_1^{k-1}, x_2^{k-1}, \dots, x_m^{k-1}),$$

which we replace by the system of all possible plans of the form

$$(*, *, \dots, *, x_{q-1}, x_q^{k-1}, \dots, x_p^{k-1}, \dots, x_m^{k-1}),$$

where $x_{q-1} \in Y_{q-1}$, $x_{q-1} \neq x_{q-1}^{k-1}$, $q = 2, 3, \dots, p$, and

$$\sum_{i=1}^{q-2} \max_{1 \le k \le s_i} a_{ik} \ge c - \sum_{i=q}^{m} x_i^{k-1} - x_{q-1}.$$

Next we find such a plan X_k that

$$\bar{F}(X_k) = \min_{X \in W_k} \bar{F}(X),$$

and pass to the next step.

Introduce the notation

$$\bar{M}_k = \bigcup_{X \in W_k} \bar{M}(X).$$

Lemma 2. $\bar{M}_1 = \bar{M},\ \bar{M}_k = \bar{M}_{k-1} \setminus X_{k-1},\ k = 2, 3, \dots.$

From Lemmas 1 and 2 it follows that

Theorem 1. The algorithm constructs such a sequence of plans $X_1, X_2, \dots$ of problem $\bar{A}$ that

$$\bar{F}(X_k) = \min_{X \in \bar{M} \setminus \{X_1, X_2, \dots, X_{k-1}\}} \bar{F}(X), \qquad k = 1, 2, \dots$$

If, from the constructed sequence of plans $X_1, X_2, \dots$ of problem $\bar{A}$, we remove the plans that do not belong to the set G, then for the remaining sequence of plans $X_1', X_2', \dots$ of problem A the following holds.

Theorem 2. If there exists a number k such that

$$F(X_k') \ge \min\{F(X_1'), F(X_2'), \dots, F(X_{k-1}')\} = F(X^*),$$

then X^* is an optimal plan of problem A.

Theorem 2 specifies an algorithm ψ for solving problem A. The desired algorithm ψ is described as follows: the k-th step consists in constructing, with the aid of algorithm φ, the plan X_k and checking whether it belongs to the set G. If not, we pass to the next step of algorithm ψ. If yes, we check whether the condition of Theorem 2 is satisfied. If not, we pass to the next step. If yes, the process terminates and X^* is an optimal plan.

The method presented is applicable to solving facility location problems ([2–4]), integer programming problems ([5], [6]), reliability problems for a complex system ([7]), and the traveling-salesman problem.

Belorussian State University
named after V. I. Lenin
Minsk

Received
7 I 1969

REFERENCES

1. R. Bellman, *Dynamic Programming*, IL, 1960.
2. V. A. Emelichev, V. I. Komlik, Dokl. AN BSSR, 10, No. 10 (1966).
3. V. A. Emelichev, V. I. Komlik, Dokl. AN BSSR, 12, No. 10 (1968).
4. V. I. Komlik, Izv. AN BSSR, Ser. Phys.-Math. Sci., No. 4 (1967).
5. V. I. Komlik, V. A. Emelichev, Dokl. AN BSSR, 11, No. 12 (1967).
6. V. I. Komlik, V. A. Emelichev, Izv. AN BSSR, Ser. Phys.-Math. Sci., No. 4 (1968).
7. V. A. Emelichev, V. I. Komlik, Izv. AN BSSR, Ser. Phys.-Math. Sci., No. 5 (1968).

Note: Figure translations are in progress. See original paper for figures.