A BICOMPACTUM WITH THE FIRST AXIOM OF COUNTABILITY WHOSE DIMENSIONS \(\operatorname{ind}\) AND \(\dim\) DO NOT COINCIDE

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Abstract

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A BICOMPACTUM WITH THE FIRST AXIOM OF COUNTABILITY WHOSE DIMENSIONS ind AND dim DO NOT COINCIDE

(Presented by Academician P. S. Aleksandrov, 19 XI 1968)

P. S. Aleksandrov posed the question of the coincidence of the dimensions ind and dim for bicompacta. A. L. Lunts (^{4}) and O. V. Lokutsievskii (^{3}) constructed bicompacta with noncoinciding dimensions ind and dim. In both examples the essential point was the failure of the first axiom of countability. Recently V. V. Fedorchuk (^{6}) constructed a bicompactum X with the first axiom of countability with $3 \leq \operatorname{ind} X \leq 4$, $\dim X = 2$. In the present note we shall give an example of a bicompactum X with $\operatorname{ind} X = 2$, $\dim X = 1$, represented in the form of the union of bicompacta Y_1, Y_2 with $\operatorname{ind} Y_1 = \operatorname{ind} Y_2 = 1$.

Let Z be the lexicographically ordered square, i.e. the set of all ordered pairs $\{a_1, a_2\}$, where $a_1, a_2 \in I = [0,1]$, between which an order relation $<$ is defined, namely, it is assumed that $\{a_1, a_2\} < \{\beta_1, \beta_2\}$ if $a_1 < \beta_1$, or $a_1 = \beta_1$ and $a_2 < \beta_2$. The space Z in the order topology is, obviously, a connected bicompactum with the first axiom of countability, and all its dimensions are equal to 1.

Let K be the Cantor perfect set lying in the interval. If we identify pairwise the endpoints of adjacent intervals, then we obtain something homeomorphic to an interval; moreover, taking arbitrarily some countable everywhere dense set lying in the interval and not containing endpoints, we can construct this homeomorphism so that it establishes a one-to-one correspondence between this set and the identified pairs and preserves, up to the identified points, the order of the Cantor perfect set. We shall need two such disjoint sets Q_1 and Q_2. Denote by λ_1 and λ_2 the corresponding maps of the Cantor perfect set onto the interval.

The mapping $\lambda_i : K \to I$ induces a mapping $\lambda_i^* : Z \times K \to Z \times I$ by the formula $\lambda_i^*((z,k)) = (z, \lambda_i(k))$.

In the space Z there lies, as a closed set, the subspace S of points of the form $\{a, 0\}$ or $\{a, 1\}$. (This subspace is nothing other than "two arrows" (^{1}).)

Identify points of the product $Z \times K$ if: a) they belong to the (closed) subspace $S \times K$, and b) their images under λ_i^* coincide. The quotient-space Y_i obtained under this factorization φ_i is, obviously, a bicompactum.

The mapping λ_i^* can be represented (in a unique way) as the superposition of the quotient mapping φ_i and the mapping $\lambda_i^{**} : Y_i \to Z \times I$. The correspondence $\lambda_i^{**-1} : Z \times I \to Y_i$ is one-to-one on $S \times I$ and, as is easy to see, is a topological embedding of the subspace $S \times I$ into Y_i. We shall say, simply, that $S \times I$ lies in Y_i.

Since the space Y_i contains arbitrarily many intervals, all its dimensions are at least 1. In the product $Z \times K$ we can choose a neighborhood U of the preimage $\varphi_i^{-1}(y)$ of a point $y \in Y_i$ in the form of a square, two of whose sides are empty and the other two have the form $\{z'\} \times K'$ and $\{z''\} \times K'$, where the set K' is homeomorphic to the Cantor perfect set, and the points z' and z'' do not belong to S. Then the boundary of the (open) set V, consisting of those points of the space Y_i whose complete preimages lie ···

in U, is the union of four (disjoint) zero-dimensional compacta, which are Cantor-set intervals or "two arrows." It is easy to see that every point of the space Y_i has arbitrarily small neighborhoods of this kind and, consequently, $\operatorname{ind} Y_i = 1$. (We note that $\operatorname{Ind} Y_i = 1$, since the union of any finite number of neighborhoods of the indicated kind has a zero-dimensional boundary.) Moreover, it is true (see (2)) that $\operatorname{ind} Y_i \geq \dim Y_i$, and, consequently, $\dim Y_i = 1$.

We have two spaces Y_1 and Y_2. In each of them a subspace $S \times I$ is embedded. In the disjoint sum $Y_1 \cup Y_2$ we identify the two copies $S \times I$ into one. The quotient mapping Φ obtained in this way gives a quotient space X, which is obviously a bicompactum. We shall not distinguish between the spaces Y_1 and Y_2 and their images under the quotient mapping (which, as is easy to see, is a homeomorphic embedding on each of them).

The mappings λ_1^{**} and λ_2^{**} coincide on the set $Y_1 \cap Y_2 = S \times I$, and therefore there exists a (continuous) mapping $\lambda : X \to Z \times I$, coinciding on Y_i with λ_i^{**}. Since the inverse image of each point of the set $S \times I \subset Z \times I$ (under the mapping λ) consists of exactly one point of the set $S \times I \subset X$, the full inverse images of open subsets of $Z \times I$ form a base at the points $S \times I \subset X$.*

Since the space X is represented as the union of two one-dimensional bicompacta, it follows (see (2)) that $\dim X = 1$.

We shall show that $\operatorname{ind} X \geq 2$. For this it is enough to verify that any open subset of X intersecting $S \times I$ and not intersecting $\lambda^{-1}(Z \times \{0, 1\})$ contains in its boundary some interval. Let V be such a set, $x = (s, r) \in V \cap (S \times I)$. Then there exists an open set $U = (z_1, z_2) \times (r_1, r_2) \subset Z \times I$ such that $\lambda^{-1}(U)$ contains the point x and lies in V together with its closure. We note that the interval (z_1, z_2), containing the point s, must contain an uncountable set of points of the set S. Let $V_i = V \cap Y_i$ $(i = 1, 2)$; let π_1 and π_2 be the projections of $Z \times I$ onto the first and second factors, respectively, and let $f_i = \pi_i \lambda$ $(i = 1, 2)$. To a point $z \in (z_1, z_2)$ assign the number

$$g_i(z) = \sup\{f_2(y) : y \in V_i \cap f_1^{-1}(z)\}.$$

We observe that if $g_i(z) \notin Q_i$, then the (unique) point

$$\xi \in f_1^{-1}(z) \cap f_2^{-1}(g_i(z)) \cap Y_i$$

belongs to the boundary of the set V_i.

Let us see what happens if the number of intervals $I_\alpha = [\{a,0\},\{a,1\}] \subset (z_1, z_2)$ on which the function $g_i(z)$ is nonconstant is uncountable. In each such interval I_α there is a pair of points $z_\alpha, z'_\alpha \in I_\alpha$ for which $g_i(z_\alpha) > g_i(z'_\alpha)$. From the definition of $g_i(z)$ it follows easily that there exists an interval G_α in the interval I_α such that

$$Y_i \cap f_2^{-1}(G_\alpha) \cap f_2^{-1}(z_\alpha) \subset V_i, \qquad Y_i \cap f_2^{-1}(G_\alpha) \cap f_1^{-1}(z'_\alpha) \subseteq Y_i \setminus V_i.$$

We have assumed that the number of intervals I_α on which the function $g_i(z)$ is nonconstant is uncountable, and therefore, by virtue of the fact that the segment has a countable base, there exists an interval $H \subseteq I$ contained in uncountably many intervals G_α. The uncountable set $\{\{a,0\} : G_\alpha \supseteq H\}$ must have a limit point $z_0 \in S \cap (z_1, z_2)$. In this case, as is easy to see, the point z_0 is also a limit point for the points $\{z_\alpha, z'_\alpha : G_\alpha \supseteq H\}$, and the set $\{z_0\} \times H \subseteq S \times I \subseteq Y_i$ consists of limit points both for V_i and for $Y_i \setminus V_i$, and, consequently, the boundary of the set V contains an interval.

Let us now see what happens if the functions g_1 and g_2 are constant everywhere, except possibly for a countable family of intervals of the form I_α. In this case there exists an interval $I_\alpha = [\{a,0\},\{a,1\}]$ on which they are both constant. But, by their definition, they coincide at its endpoints, and hence they coincide on the whole interval. Let their common value be R. Thus since $Q_1 \cap Q_2 = \emptyset$, at least one of these sets does not contain R. Let $R \not\subset Q_i$. In this case the set $Y_i \cap f_2^{-1}(R) \cap f_1^{-1}(I_\alpha)$ is homeomorphic to an interval and consists of points lying on the boundary of the set V_i. Thus, in this case as well the boundary of the set V contains an interval.

Thus we have proved that $\operatorname{ind} X \geq 2$. But sets of the form $\lambda^{-1}((z_1, z_2) \times (r_1, r_2))$, having a one-dimensional boundary, form a base at the points belonging to $S \times I$, and at all other points there exist arbitrarily small neighborhoods with zero-dimensional boundary. Moreover, finite unions of all such neighborhoods have one-dimensional boundary. Therefore $2 \leq \operatorname{ind} X \leq \operatorname{Ind} X \leq 2$, i.e., all inductive dimensions are equal to 2.

Remark. The space constructed is not separable. But every bicompactum of cardinality not exceeding C (under the assumption that $C = \aleph_1$) is the space of a decomposition of the remainder $\beta N \setminus N$ in the Stone–Čech extension of a

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* For any closed mapping (and every continuous mapping of a bicompactum is such), the full inverse images of open sets form a base of neighborhoods of the full inverse images of closed sets.

countable discrete space N (see (5)). If in the space βN, leaving the set N fixed, one identifies the points of the remainder so as to obtain the bicompactum Y_i, then we obtain a certain bicompactum Y_i' containing a topological image of the bicompactum Y_i, and the everywhere dense set $Y_i' \setminus Y_i$ consists of a countable number of isolated points. It is not hard to see that in the separable space Y_i' the first axiom of countability is satisfied (see (7)) and that $\dim Y_i' = \operatorname{ind} Y_i' = \operatorname{Ind} Y_i' = 1$. Gluing the necessary parts of the subspaces $Y_1 \subset Y'$ and $Y_2 \subset Y_2'$, we obtain a separable bicompactum X' satisfying the first axiom of countability and with $\dim X' = 1$, $\operatorname{ind} X' = \operatorname{Ind} X' = 2$.

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