ELLIPTIC PROBLEMS OF S. L. SOBOLEV TYPE FOR SUBMANIFOLDS WITH POINT SINGULARITIES

MATHEMATICS

1969

SovietRxiv

Abstract

Full Text

UDC 517.9

MATHEMATICS

B. STERNIN

ELLIPTIC PROBLEMS OF S. L. SOBOLEV TYPE FOR SUBMANIFOLDS WITH POINT SINGULARITIES

(Presented by Academician S. L. Sobolev on 28 V 1968)

1. Introduction. By elliptic problems of S. L. Sobolev type we mean problems for elliptic operators, defined on a manifold X, in which the boundary conditions are posed on submanifolds Y that are not the boundary of the principal manifold. This will always be the case, for example, if the manifold Y (as a submanifold of X) has codimension greater than 1; however, this condition is not necessary.

Problems of this type were first considered by S. L. Sobolev (^{1}). By the variational method he proved the unique solvability of the Dirichlet problem for the polyharmonic equation $\Delta^m u = 0$ in the space $u \in W_2^{(m)}$ for admissible (in the terminology of the book (^{1})) boundary functions.

In the paper (^{2}) a theory was developed for general elliptic pseudodifferential problems of S. L. Sobolev type in the case when the boundary submanifolds were smooth closed submanifolds without boundary. In the note (^{3}) the same result was extended to the case when the boundary submanifolds have boundary.

In the present paper a theory is constructed for elliptic problems of S. L. Sobolev type for submanifolds that may have point singularities. This means that the boundary submanifolds must be smooth everywhere except for a finite number of points. The nature of the singularities at these points must be "conical." Here are typical examples. In them the term manifold will always denote the boundary manifold.

1) The manifold consists of one or several points.

2) The manifold is the intersection* of a finite number of smooth manifolds (with boundary or without boundary), and the intersection-manifold consists of one or several points.

3) The manifold is a cone consisting of an arbitrary number of half-lines.

2. Method and bibliographical remarks. The method used in the present work consists in reducing the initial problem to an analytic family of Sobolev-type problems for smooth boundary submanifolds. This reduction is based on the use of the Mellin transform instead of the Fourier transform, which is well adapted only to the smooth case. This method was first applied by G. I. Eskin (^{4}) to the study of elliptic boundary-value problems in plane domains with angular points. Later V. A. Kondrat' ev (^{5}) used this method to study multidimensional elliptic boundary-value problems in domains with conical points.

3. Description of the singularity. Let X be a smooth manifold without boundary, and let $Y \subset X$ be a submanifold (with boundary or without boundary), embedded in the manifold X in such a way that the singular* submanifold $Z \subset Y$ consists of a finite number of points. If the manifold Y consists of a single point, then we shall also regard this point as singular. Let us give another description of the singularities of the manifold Y. Let $z \in Z$ be an arbitrary point. Then there must exist so small an $\varepsilon > 0$ that the trace of the manifold Y on the sphere of radius ε with center at the point z is a finite number of smooth closed manifolds with boundary or without boundary. This is an exact description of the situation with which we shall deal.

4. Functional spaces. Let R^{N+1} be the Euclidean space of $N+1$ dimensions with coordinates $(x^1, \dots, x^{N+1})$. Further, let $\omega_i = \omega_i(x^1, \dots, x^{N+1})$, $i = 1, \dots, N$, be some system of coordinates from a smooth atlas of local charts on the sphere S^N. We shall define a mapping of the space R^{N+1} onto a semi-infinite cylinder by setting

$$r = \sqrt{x^{1\,2} + \cdots + x^{N+1\,2}}, \tag{1}$$

$$\omega_i = \omega_i(x^1, \dots, x^{N+1}), \qquad i = 1, \dots, N. \tag{2}$$

In the new coordinates any functional space may be interpreted as a space of functions defined on the half-axis $0 \le r < +\infty$ and taking values in a space of functions on the sphere S^N. Let now s and a be arbitrary real numbers. Denote by Γ^s_α the completion of the space of smooth finite functions on the half-axis with respect to the norm

$$\|f\|_{s,\alpha} = \int_{\alpha - s + (N+1)/2 - i\infty}^{\alpha - s + (N+1)/2 + i\infty} \left\| \left(Z + \sqrt{1 - \Delta}\right)^s \hat{f}(Z) \right\| dZ.$$

---

* We shall always assume that all manifolds are embedded in Euclidean space of sufficiently high dimension.

Here Δ is the Laplace operator on the sphere S^N, constructed by means of a certain Riemannian metric, which from this moment on is regarded as fixed, and $\hat{f}(Z)$ is the Mellin transform of the function f:

$$\hat{f}(Z) = \int_0^\infty r^{Z-1} f(r)\, dr.$$

We note that for nonnegative integer values of s the space Γ^s_α coincides with the space of functions f for which the integral

$$\sum_{k=0}^{s} \int_{R_+^{N+1}} r^{2\alpha-2s} \left| r^k D^k f \right|^2 \, dx$$

is finite; this follows directly from Parseval' s formula for the Mellin transform (see, for example, (6)).

5. The local situation. Let D be an elliptic differential expression on the manifold X. We shall study the solvability of $Du \equiv f$, i.e., of the equation $Du = f$ outside Y. To this problem we shall apply the *a posteriori* principle of locality, so that the main emphasis falls on the study of the local situation. With this in mind, we shall assume that the manifold X is realized as an $(N+1)$-dimensional Euclidean space.

In the space $X = R^{N+1}$ there is embedded a submanifold Y of the singular form discussed in §3. Let now $D = \sum a_\alpha D^\alpha$ be a homogeneous differential expression in the space R^{N+1} with constant coefficients.

* By singular we mean, by definition, a submanifold Z of the manifold Y, consisting of such a set of points that no neighborhood in Y of an arbitrary point of this set can be mapped diffeomorphically onto a domain of Euclidean space.

We shall study the congruence

$$Du \equiv f \pmod{Y} \tag{3}$$

in the space of distributions $u \in \Gamma^s_\alpha$, if the right-hand side f belongs to the space Γ^t_α. Let $Y_{\nu_1}, \dots, Y_{\nu_l}$ be the components of the manifold Y with codimensions (in X) $\nu_1, \dots, \nu_l$ (it is not excluded that the manifold Y has several components of the same codimension). For each number ν_i we define an integer nonnegative number $\chi_i = \chi(\nu_i)$, putting

$$\chi_i = \begin{cases} [-t - \nu_i/2], & \text{if } -t-\nu_i/2 \text{ is not an integer}, \\ -t - \nu_i/2 - 1, & \text{if } -t-\nu_i/2 \text{ is an integer}. \end{cases}$$

We now prescribe, on each component of codimension ν_i, differential boundary conditions

$$B_{j_i} = \sum a_\alpha^{j_i} D^\alpha\big|_{Y_i} = g_{j_i}, \qquad |j_i| \le \chi_i, \quad i \le l. \tag{4}$$

Here $j_i = (j_{1_i}, \dots, j_{\nu_i})$, $|j_i| = j_{1_i} + \dots + j_{\nu_i}$. We shall require that at each nonsingular point of the manifold Y an algebraic condition be satisfied which guarantees the normal solvability of the corresponding problem in the nonsingular case (^{2},^{3}).

We now make the transformation (1), (2) in the congruence (3) and equation (4). Then we obtain a system of differential equations on the half-infinite cylinder

$$\sum_{\alpha+\beta \le s-t} \frac{a_{\alpha\beta}}{r^\alpha} \frac{\partial^\alpha}{\partial r^\alpha} D_\omega^\beta u \equiv f(r,\omega) \quad (\operatorname{mod} \widetilde{Y}),$$

$$\sum_{\alpha+\beta \le s-t_{j_i}} \frac{b_{\alpha\beta}}{r^\alpha} \frac{\partial^\alpha}{\partial r^\alpha} D_\omega^\beta u\big|_{\widetilde{Y}} = g_{j_i}(r,\omega).$$

Here $\widetilde{Y}$ denotes the image of the manifold Y under the mapping (1), (2). The Mellin transform with respect to the variable r reduces the system written above to a family of Sobolev-type problems on the N-dimensional sphere S^N, which we shall conventionally write as

$$D_z(\omega)\hat{u}_z(\omega) \equiv \hat{F}_z(\omega) \quad (\operatorname{mod} \widetilde{Y}_\omega), \tag{5}$$

$$B_z(\omega)\hat{u}_z(\omega) = \hat{G}_z(\omega) \quad \text{on } \widetilde{Y}_\omega. \tag{6}$$

Here $\hat{F}_z(\omega)$ denotes the image, under the Mellin transform, of the distribution $F = r^{s-t} f(r,\omega)$, and $\hat{G}_z(\omega)$ the images of the distributions $r^{s-t_{j_i}} g_{j_i}(r,\omega)$.

Our main requirement is that problem (5), (6) be uniquely solvable. Let us try to give this assertion a more algebraic form. Suppose, for simplicity, that all the manifolds $\widetilde{Y}_\omega$ are closed. Then the family of problems (5), (6) can be realized as an analytic family of operators

$$\mathfrak{A}(Z) : \Gamma^s(S^N) \to \Gamma^t(S^N) / \operatorname{mod} \widetilde{Y}_\omega \oplus \bigoplus_{j_i} \Gamma_{j_i}^{t_i-\nu_i/2}(\widetilde{Y}_\omega). \tag{7}$$

We require that, for each manifold $\widetilde{Y}_\omega$, the algebraic conditions formulated in (^{2}) be satisfied. In this case the family (7) is a family of Fredholm operators. Moreover, the following stronger theorem is valid (cf. (^{7})):

Theorem 1. For any positive number $h > 0$ there exists a positive number $k = k(h) > 0$ such that for all z with $|\operatorname{Re} z| \le h$ and $|\operatorname{Im} z| \ge k$ the operator

$\mathfrak{A}(z)$ is invertible on the left and on the right; moreover, the a priori estimate holds:

$$\left\|(z+\sqrt{1-\Delta})^s\hat{u}\right\| \leq \text{const}\left(\left\|(z+\sqrt{1-\Delta})^t\hat{f}\right\| + \sum_{j_i}\left\|(z+\sqrt{1-\Delta'})^{t_{j_i}-\nu_i/2}\widehat{G}\right\|\right).$$

Here Δ' denotes the Laplace operator on the submanifold Y_ω, and $\|\cdot\|$ denotes the norms in L_2. It follows from this theorem that there exists an operator-valued function $R(z)$, meromorphically depending on the parameter z and inverting the operator $\mathfrak{A}(z)$ at regular points.

Now applying the Mellin transform inversely, we obtain that the solution $u(r,\omega)$ of problem (2), (3) is given by the formula

$$u = \int_{a-s+(N+1)/2-i\infty}^{a-s+(N+1)/2+i\infty} r^{-z}R(z)[\hat{F},\widehat{G}]\,dz \tag{8}$$

in the case when there are no poles on the line $\operatorname{Re} z = a-s+(N+1)/2$.

Representation (8) makes it possible to establish the following theorem.

Theorem 2. Suppose the function $r^{-z}R(z)$ is regular on the line

$$\operatorname{Re} z = a-s+(N+1)/2.$$

Then the problem

$$Du \equiv f(\operatorname{mod} Y), \qquad Bu = g \quad (\text{on } Y)$$

is uniquely solvable in the spaces

$$u \in \Gamma_a^s, \qquad f \in \Gamma_a^t, \qquad g_{j_i} \in \Gamma^{t_{j_i}-\nu_i/2},$$

and the inequality holds:

$$\|u\|_{s,a} \leq \text{const}\left(\|f\|_{t,a} + \sum_{j_i}\|g_{j_i}\|_{t_{j_i}-\nu_i/2}\right).$$

From this theorem, in the usual way, the normal solvability of the problem

$$Du \equiv f(\operatorname{mod} Y), \qquad Bu = g \quad (\text{on } Y),$$

where D and B are differential expressions with smooth complex-valued coefficients, is proved, and the corresponding a priori estimate is established.

Further, we show that in each finite strip $h_1 \leq \operatorname{Re} z \leq h_2$ $(h_1 \neq -\infty,\ h_2 \neq +\infty)$ the function $r^{-z}R(z)$ has only finitely many poles, so that, using the residue theorem, one can obtain the asymptotics of the solution near the singular point.

Institute for Problems in Mechanics
Academy of Sciences of the USSR

Received
15 V 1968

CITED LITERATURE

1. S. L. Sobolev, *Some Applications of Functional Analysis in Mathematical Physics*, Novosibirsk, 1962.
2. B. Yu. Sternin, Tr. Mosc. Math. Soc., **15**, 346 (1966).
3. B. Yu. Sternin, DAN, **183**, No. 2 (1968).
4. V. A. Kondrat' ev, Tr. Mosc. Math. Soc., **16** (1967).
5. G. I. Eskin, UMN, **18**, issue 3 (1963).
6. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, Moscow-Leningrad, 1948.
7. M. S. Agranovich, M. I. Vishik, UMN, **19**, issue 3 (1964).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.