ON A PROBLEM OF F. HALL

MATHEMATICS

1969

SovietRxiv

Abstract

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UDC 519.45

MATHEMATICS

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ON A PROBLEM OF F. HALL

(Presented by Academician A. N. Tikhonov on 4 VII 1969)

One of the interesting and important problems in the theory of representations of groups is the problem of studying the properties of groups as a function of the properties of their representations. In particular, the following question arises: what properties must a group possess in order that all irreducible representations over an arbitrary field be finite-dimensional? It is known ([1–3]) that, in order that all irreducible representations of a group Γ be finite-dimensional, it is sufficient that Γ have an abelian normal divisor of finite index and a finite number of generators. F. Hall ([3]) proved that every irreducible representation of a finitely generated nilpotent group over an absolutely algebraic field of prime characteristic p is finite-dimensional. On the other hand, in the same paper it is shown that if a polycyclic group Γ does not have an abelian normal divisor of finite index and a finite number of generators, and the field P is distinct from an absolutely algebraic field of prime characteristic p, then Γ has an infinite-dimensional representation over the field P.

The present note is devoted to the proof of a theorem asserting that every irreducible representation of a polycyclic group over an absolutely algebraic field of prime characteristic p is finite-dimensional. This theorem is an answer to the question posed by F. Hall in ([3]).

We shall use the following notation: Γ is a certain group; $\gamma, \sigma, \varphi, \ldots$ are elements of the group Γ; P is an absolutely algebraic field of prime characteristic p; G is a vector space over the field P; $x, y, z, \ldots$ are elements of G; $[\Gamma]$ is the group algebra of the group Γ over the field P; $R(\Gamma)$ is the locally nilpotent radical of the group Γ; $Z(R)$ is the center of the group $R(\Gamma)$; (G, Γ) is a representation of the group Γ by automorphisms of the vector space G over the field P.

Let Γ be a polycyclic group, i.e. a group possessing a finite normal series with cyclic factors:

$$\{e\} = \Gamma_0 \subset \Gamma_1 \subset \ldots \subset \Gamma_n = \Gamma. \quad (1)$$

We note that the group Γ is polycyclic if and only if Γ is a solvable group satisfying the maximal condition for subgroups ([4]).

Let I be a commutative domain of principal ideals which is not a field, and let $\omega = \{p_\alpha\}$ $(\alpha \in \Lambda)$ be the set of all prime elements in I. We shall say that an I-module V belongs to the class of I-modules $\mathfrak{A}(I, \omega', k)$ $(\omega' \subset \omega,\ k$ a natural number) if and only if in V there is a free I-submodule V_0 such that V/V_0 is an ω'-periodic I-module and the generator g of the order ideal $\mathfrak{D}(v+V_0)$ $(v \in V,\ \mathfrak{D}(v+V_0) = \{h \in I :\ vh \in V_0\} = gI)$ of an arbitrary element $v + V_0 \in V/V_0$ can be represented in the form

$$g = \prod_{\alpha \in \Lambda} p_\alpha^{n_\alpha} \quad (\alpha \in \Lambda),$$

where only a finite number of the n_α are distinct from zero and all $n_\alpha \leq k$.

An I-module V is called ω'-divisible $(\omega' \subset \omega)$ if for any $v_0 \in V$, $\lambda \in I$ the equation $v\lambda = v_0$ has a solution in V. If V_0 is a submodule of the I-module V, then we shall call it servile if from the fact that the equation $v\lambda = v_0$ $(v_0 \in V$, $\lambda \in I)$ has a solution in V it follows that it has a solution in V_0.

Let $\sigma \in Z(R)$, and let (G, Γ) be some representation of the group Γ, $I = P[x]$. Then it is clear that G can be regarded as an I-module if the representation of I relative to G is given by the mapping $x \to \sigma$. For convenience, in this case we shall say that G is an I_σ-module.

Lemma 1. *Let (G, Γ) be a cyclic representation of the polycyclic group Γ. Then every minimal servile I-submodule containing an element $x \in G$ belongs to the class $\mathfrak{A}(I_\sigma, \omega', k)$, where ω' is a subset of ω.*

Let (G, Γ) be some representation of the polycyclic group Γ. Denote by

$$G(\sigma_0, \lambda_0) = \{x \in G :\ x = y(\sigma_0 - \lambda_0 e),\ y \in G\}$$

$(\sigma_0 \in Z(R),\ \lambda_0 \in P)$, and by $G(\sigma_1, \sigma_2, \dots, \sigma_k, \lambda_1, \lambda_2, \dots, \lambda_k)$ the linear span of the subspaces

$$G(\sigma_1, \lambda_1),\ G(\sigma_2, \lambda_2), \dots,\ G(\sigma_k, \lambda_k) \quad (\sigma_i \in Z(R),\ \lambda_i \in P).$$

It is clear that each $G(\sigma_i, \lambda_i)$ is a $[Z(R)]$-submodule in G.

Lemma 2. *Let (G, Γ) be a cyclic representation of the polycyclic group Γ, $\sigma_1, \sigma_2, \dots, \sigma_m \in Z(R)$, $\lambda_1, \lambda_2, \dots, \lambda_m \in P$. Then every minimal servile $[\sigma]$-submodule of the $[Z(R)]$-module*

$$G/G(\sigma_1, \sigma_2, \dots, \sigma_m, \lambda_1, \lambda_2, \dots, \lambda_m),$$

containing the element x_0, belongs to the class $\mathfrak{A}(I_\sigma, \omega', k)$, where $\omega' \subset \omega$ is a subset of primes.

From the preceding two lemmas the following proposition follows.

Main Lemma. *Let (G, Γ) be an irreducible representation of a polycyclic group over an absolutely algebraic field of prime characteristic p. Then in G there exists a maximal proper subspace invariant with respect to $Z(R)$.*

Theorem. *Every irreducible representation of a polycyclic group Γ by automorphisms of a vector space G over an absolutely algebraic field of prime characteristic p is finite-dimensional.*

Proof.* Let (G, Γ) be an exact irreducible representation of the group Γ. Then for any x we have the equality $x[\Gamma] = G$. Let

$$Q = \{g \in [\Gamma] : \ xg = 0\}.$$

It is easy to see that Q is a right ideal in $[\Gamma]$ and that the $[\Gamma]$-modules G and $[\Gamma]/Q$ are isomorphic. According to the main lemma, in G there is a maximal $Z(R)$-invariant subspace H. Since $Z(R)$ is an abelian group, the quotient space G/H is one-dimensional (5), and this means that for each element $\sigma \in Z(R)$ there is such a $\lambda \in P$ that in the quotient G/H the automorphism σ induces the automorphism λe (e is the identity automorphism). From the fact that P is an absolutely algebraic field of prime characteristic p, it follows that for any σ there is such a natural number n that σ^n induces the identity automorphism in the vector space G/H. Since $Z(R)$ is an abelian group with a finite number of generators, there is such a k that σ^k lies in the kernel of the representation $(G/H, Z(R))$ for any $\sigma \in Z(R)$. Denote by $Z^k(R)$ the subgroup in $Z(R)$ generated by the elements of the form σ^k, where $\sigma \in Z(R)$. It is easy to see that $Z^k(R)$ is a characteristic subgroup in $Z(R)$ and, consequently, a normal divisor of the group Γ. Taking into account that (G, Γ) is an irreducible representation of the group Γ, we obtain that, by Remak' s theorem, G is the direct sum of the spaces $H\gamma$ (γ ranges over a set of representatives of all adjacent classes of the group Γ with respect to the subgroup $Z(R)$). Since the group $Z^k(R)$ lies in the kernel of the representation $(G/H\gamma, Z(R))$ for any representative γ, it lies in the kernel of the repre–

* It is enough to consider the case of an algebraically closed field P.

representation (G, Γ). Therefore $Z(R)$ is a finite group, and then the group Γ is finite. Consequently, the vector space G is finite-dimensional. The theorem is proved.

Corollary. A polycyclic group Γ has a faithful irreducible representation over a field P if and only if it is finite.

Received
15 II 1968

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Note: Figure translations are in progress. See original paper for figures.