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Abstract

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AN APPROXIMATION THEOREM OF A. F. TIMAN FOR FUNCTIONS HAVING A CONTINUOUS DERIVATIVE OF FRACTIONAL ORDER

(Presented by Academician S. L. Sobolev on March 17, 1969)

A. F. Timan (^{1}) established the following theorem:

If a function $f(x)$, defined on the interval $[-1,1]$, has there an r-th continuous derivative (r an integer), then for every $n \geq r$ there exists an ordinary polynomial $P_n(x)$ of degree $\leq n$ such that

$$|f(x) - P_n(x)| \leq C_r \left(\frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2} \right)^r \omega \left(\frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2} \right), \tag{1}$$

where C_r is a constant independent of n and f;

$$\omega(t) = \omega(f^{(r)}, t) = \sup_{|x_1 - x_2| \leq t} |f^{(r)}(x_1) - f^{(r)}(x_2)| \quad (x_1, x_2 \in [-1,1]).$$

Later I. E. Gopengauz (^{2}) and S. A. Telyakovskii (^{3}) showed that inequality (1) can be replaced by the inequality

$$|f(x) - P_n(x)| \leq C_r \left(\frac{\sqrt{1-x^2}}{n} \right)^r \omega \left(\frac{\sqrt{1-x^2}}{n} \right). \tag{2}$$

In the present note we give a generalization of these results to the case where the function $f(x)$ has a continuous derivative of arbitrary fractional order r.

Theorem. *For every function $f(x)$, defined on $[0,1]$, for which there exists a continuous derivative of fractional order r ($r = r' + \alpha$, r' an integer, $0 < \alpha < 1$), for every natural $n \geq r - 1$ one can specify an algebraic polynomial $P_n(x)$ of degree $\leq n$ such that*

$$|f(x) - P_n(x) - cx^r| \leq C_r \left(\frac{\sqrt{x(1-x)}}{n} \right)^r \omega \left(f^{(r)}; \frac{\sqrt{x(1-x)}}{n} \right), \tag{3}$$

where C_r is a constant independent of n and f; c is a constant depending only on f and r.

The proof of this theorem is based on the following auxiliary proposition.

Lemma. *If*

$$F_\alpha(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \varphi(t)\, dt \quad (0 < \alpha < 1), \tag{4}$$

where $\varphi(t)$ *is some continuous function and* $\varphi(0) = 0$, *then*

$$\omega_2(F_\alpha, h) \leq C(\alpha) h^\alpha \omega(\varphi, h) \quad (h > 0), \tag{5}$$

where

$$\omega_2(F_\alpha, h) = \sup_{|x_1 - x_2| \leq h} \left| F_\alpha(x_1) - 2F_\alpha\left(\frac{x_1 + x_2}{2}\right) + F_\alpha(x_2) \right|, \quad x_1, x_2 \in [0,1].$$

Inequality (4), after extending the function $\varphi(t)$ so that $\varphi(t) = 0$ for $-\infty \leq t \leq 0$, is verified directly.

Representing now, in accordance with the known definition of the fractional derivative (see (4), § 9.8), the function $f(x)$ by the equality

$$f(x) = \sum_{k=0}^{r'-1} c_k x^k + cx^r + \frac{1}{(r'-1)!} \int_0^x (x-t)^{r'-1} F_\alpha(t)\, dt,$$

where $F_\alpha(x)$ is expressed by equality (4), and $\varphi(t) = f^{(r)}(t) - f^{(r)}(0)$, and applying to the function

$$\Phi(x) = \frac{1}{(r'-1)!} \int_0^x (x-t)^{r'-1} F_\alpha(t)\, dt = f(x) - \sum_{k=0}^{r'-1} c_{kx}^k - cx^r$$

the theorem of I. E. Gopengauz (^{2}), we find a polynomial $Q_n(x)$ of degree $\leq n$ ($n \geq r$) such that

$$|\Phi(x) - Q_n(x)| \leq C_{r'} \left(\frac{\sqrt{x(1-x)}}{n} \right)^{r'} \omega_2 \left(\Phi^{(r')}, \frac{\sqrt{x(1-x)}}{n} \right).$$

If now

$$R_n(x) = Q_n(x) + \sum_{j=0}^{r'-1} c_{kx}^{k},$$

then we have

$$|f(x) - R_n(x) - cx^r| \leq C_{r'} \left(\frac{\sqrt{x(1-x)}}{n} \right)^{r'} \omega_2 \left(F_\alpha, \frac{\sqrt{x(1-x)}}{n} \right).$$

After applying the lemma to the right-hand side of the last inequality, we obtain the assertion of the theorem.

Let us note that in the left-hand side of inequality (3), when r is fractional, the term cx^r cannot be omitted. In order to see this, it suffices to consider the function $f(x) = x^r$.

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CITED LITERATURE

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[3] S. A. Telyakovskii, *Mat. sbornik*, 70 (112), No. 2 (1966).
[4] A. Zygmund, *Trigonometric Series*, 1939.

Note: Figure translations are in progress. See original paper for figures.