ASYMPTOTIC BEHAVIOR OF THE TIME OF ATTAINMENT FOR SUMS OF RANDOM VARIABLES GOVERNED BY A REGULAR SEMI-MARKOV PROCESS

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Abstract

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ASYMPTOTIC BEHAVIOR OF THE TIME OF ATTAINMENT FOR SUMS OF RANDOM VARIABLES GOVERNED BY A REGULAR SEMI-MARKOV PROCESS

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Let $\xi_1(\alpha), \xi_2(\alpha)$ be, for each $\alpha \in [0,1]$, a sequence of independent identically distributed random variables with $M\xi_1(\alpha) = a(\alpha)$ and $D\xi_1(\alpha) = b(\alpha)$. Introduce the random functional

$$\tau(\alpha, s) = \min\left(n : \sum_{k=1}^{n} \xi_k(\alpha) \geq s\right), \qquad s > 0.$$

From the results given in ([1]) it follows that in the case when:

1) $\lim\limits_{\alpha\to 1} (\text{sl})\, \xi_1(\alpha) = \xi_1(1)$ *;
2) $\lim\limits_{\alpha\to 1} a(\alpha) = a(1) = 0$;
3) $\lim\limits_{\alpha\to 1} b(\alpha) = b(1) \in (0, \infty)$.
4) there exists

$$\lim_{\alpha\to 1,\, t\to\infty} (1 + a(\alpha)\sqrt{t})^{-1} = q \in [0,1],$$

$$\lim_{\alpha\to 1,\, t\to\infty} P\left\{\frac{\tau(\alpha, w(\alpha,t))}{t} < z\right\} = \sigma(z)\sqrt{\frac{2}{\pi}} \int_{q/\sqrt{zb(1)}}^{\infty} \exp\left\{\frac{1-q}{b(1)} - \frac{(1-q)^2}{2b^2(1)v^2} - \frac{v^2}{2}\right\} dv{**},$$

where $w(\alpha, t) = \sqrt{t}/(1 - a(\alpha)\sqrt{t})$.

In the present paper this result is extended to a more general summation scheme, described below.

Let $T_1(\alpha) = \{\eta_\alpha(n),\ n = 0, 1, ...\}$ be, for each $\alpha \in [0, 1]$, a homogeneous, ergodic Markov chain with a finite or countable set of states $H = \{1, 2, \dots, m\}$, $1 \leq m \leq \infty$, transition probability matrix $\|p_{ij}(\alpha)\|_{i,j=1}^m$ and stationary distribution $q_j(\alpha) > 0,\ j \in H$, and let $T_2(\alpha) = \{(\tau(\alpha, n, i), \gamma(\alpha, n, i)),\ n \geq 1,\ i \in H\}$ be, independent of $T_1(\alpha)$, a collection of random vectors independent in the aggregate such that: a) $\tau(\alpha, n, i) \in [0, \infty)$ with probability 1, $n \geq 1$, $i \in H$; b) the distributions $(\tau(\alpha, n, i), \gamma(\alpha, n, i))$, $i \in H$, do not depend on n.

Introduce the random functionals

$$\xi(\eta_0, \alpha, t) = \sum_{k=0}^{\nu(\alpha,t)} \gamma(\alpha, k),$$

* The notation $\lim_{\alpha \to \alpha'} (\mathrm{sl})\, \xi(\alpha) = \xi(\alpha')$ denotes weak convergence of distribution functions.

** $\sigma(z) = 0$ for $z < 0$; $\sigma(z) = 1$ for $z \geq 0$.

where

$$\nu(\alpha, t) = \max\left(n : \sum_{k=0}^{n} \tau(\alpha, k) \leq t\right), \qquad \eta_0 = \eta_\alpha(0) = \mathrm{const} \in H;$$

$$\tau(\alpha, n) = 0 \text{ for } n = 0; \qquad \tau(\alpha, n) = \tau(\alpha, n, \eta_\alpha(n-1)) \text{ for } n \geq 1$$

$$\gamma(\alpha, n) = 0 \text{ for } n = 0; \qquad \gamma(\alpha, n) = \gamma(\alpha, n, \eta_\alpha(n-1)) \text{ for } n \geq 1;$$

$$\tau(\eta_0, \alpha, s) = \inf(t : \xi(\eta_0, \alpha, t) \geq s), \qquad s > 0.$$

Denote

$$\theta_0(\alpha) = \inf(t : \eta_\alpha(\nu(\alpha, t)) = i),$$

$$\theta_n(\alpha) = \inf(t : \nu(\alpha, t) > \nu(\alpha, \theta_{n-1}(\alpha)),\ \eta_\alpha(\nu(\alpha, t)) = i), \qquad n \geq 1,$$

the times of successive entries of the chain $T_1(\alpha)$ into the state $i \in H$, and

$$(\tilde{\gamma}(\alpha, n, i), \tilde{\tau}(\alpha, n, i)) = \begin{cases} (0, 0), & \text{for } n = 0, \\ \left(\theta_n(\alpha) - \theta_{n-1}(\alpha), \displaystyle\sum_{k=\nu(\alpha, \theta_{n-1}(\alpha))+1}^{\nu(\alpha, \theta_n(\alpha))} \gamma(\alpha, k)\right), & \text{for } n \geq 1. \end{cases}$$

Obviously, the random vectors $(\tilde{\gamma}(\alpha,n,i),\tilde{\tau}(\alpha,n,i))$, $n \geq 1$, are independent and identically distributed, and

$$M\exp\{\sqrt{-1}\,(s\tilde{\gamma}(\alpha,1,i)+t\tilde{\tau}(\alpha,1,i))\} =$$

$$= g_\alpha(s,t,i)\left(\sum_{\substack{k\in H\\ k\neq i}} f_\alpha(s,t,k,i)p_{ik}(\alpha)+p_{ii}(\alpha)\right),$$

where

$$g_\alpha(s,t,j) = M\exp\{\sqrt{-1}\,(s\gamma(\alpha,1,j)+t\tau(\alpha,1,j))\},\quad j\in H;$$

$$f_\alpha(s,t,j,i) = M\exp\left\{\sqrt{-1}\left(s\sum_{k=0}^{\Delta_i(\alpha)}\gamma(\alpha,k)+t\sum_{k=0}^{\Delta_i(\alpha)}\tau(\alpha,k)\right)\mid \eta_\alpha(0)=j\right\},$$

$$j\in H,\ j\neq i,$$

$$\Delta_i(\alpha)=\min(n:\eta_\alpha(n)=i),$$

and the functions $f_\alpha(s,t,j,i)$, $j\in H$, $j\neq i$, satisfy the system of linear equations

$$f_\alpha(s,t,j,i) = g_\alpha(s,t,j)\left(\sum_{\substack{k\in H\\ k\neq i}} f_\alpha(s,t,k,i)p_{jk}(\alpha)+p_{ji}(\alpha)\right),\qquad j\in H,\ j\neq i,$$

and if the corresponding moments for the random variables $\tau(\alpha,1,i)$, $\gamma(\alpha,1,i)$, $i\in H$, exist, then it is not difficult to find the quantities

$$a_1(\alpha)=q_i(\alpha)M\tilde{\gamma}(\alpha,1,i),\qquad a_2(\alpha)=q_i(\alpha)D\tilde{\gamma}(\alpha,1,i),$$

$$b(\alpha)=q_i(\alpha)M\tilde{\tau}(\alpha,1,i).$$

Theorem. Suppose that for $T_j(\alpha)$, $j=1,2$, the following conditions are satisfied:

(A): 1. $\lim\limits_{\alpha\to 1} p_{ij}(\alpha)=p_{ij}(1),\ i,j\in H.$

2. $\lim_{u\to\infty}\lim_{\alpha\to 1}\sup_{j\in H} P\{\Delta_i(\alpha) > u \mid \eta_\alpha(0) = j\} = 0.$

(B): 1. $\lim_{\alpha\to 1}(\mathrm{d})(\tau(\alpha,1,i),\gamma(\alpha,1,i)) = (\tau(1,1,i),\gamma(1,1,i)),\ i\in H.$

2. $\lim_{\alpha\to 1}\sup_{j\in H} M\tau(\alpha,1,j) < \infty.$

3. $\lim_{\alpha\to 1} M\tau(\alpha,1,i) = M\tau(1,1,i),\ i\in H.$

4. $\lim_{\alpha\to 1}\sup_{j\in H} M(\gamma,1,j)^2 < \infty.$

5. $\lim_{\alpha\to 1} M(\gamma(\alpha,1,i))^j = M(\gamma(1,1,i))^j,\ i\in H,\ j=1,2.$

(C): 1. For $\alpha > \alpha_0$, $a_1(\alpha) \geq 0$.

2. $a_1(1) = 0,\ a_2(1),\ b(1) \in (0,\infty)$.
It is not difficult to show that, under conditions (A) and (B),

$$\lim_{\alpha\to 1} a_j(\alpha) = a_j(1), \quad j=1,2, \qquad \lim_{\alpha\to 1} b(\alpha) = b(1).$$

3. There exists

$$\lim_{\alpha\to 1,\ t\to\infty}(1 + a_1(\alpha)\sqrt{t})^{-1} = g \in [0,1].$$

Then

$$\lim_{\alpha\to 1,\ t\to\infty} P\left\{\frac{\tau(\eta_0,\alpha,w(\alpha,t))}{b(1)t} < z\right\} =$$

$$= \sigma(z)\sqrt{\frac{2}{\pi}}\int_{q/\sqrt{za_2(1)}}^{\infty} \exp\left\{\frac{1-q}{a_2(1)} - \frac{(1-q)^2}{2a_2^2(1)v^2} - \frac{v^2}{2}\right\} dv,$$

where

$$w(\alpha,t) = \sqrt{t}/(1 + a_1(\alpha)\sqrt{t}).$$

Remark 1. Let $\gamma(\alpha,n,i) = \xi(\alpha,n,i,\tau(\alpha,n,i)),\ n\geq 1,\ i\in\bar{H}$, where $\{\{\xi(\alpha,n,i,t),\ t\geq 0\},\ n\geq 1,\ i\in\bar{H}\}$, independently of $T_1(\alpha)$ and $\{\tau(\alpha,n,i),\ n\geq 1,\ i\in\bar{H}\}$, is a collection of mutually independent, stochastically continuous homogeneous processes with independent increments and finite-dimensional distributions not depending on n. Then condition (B) will take the form:

1. $\lim_{\alpha\to 1}(\mathrm{d})\,\tau(\alpha,1,i) = \tau(1,1,i),\ i\in\bar{H}.$

2. $\lim\limits_{\alpha\to 1} \sup\limits_{j\in\bar{H}} M\big(\tau(\alpha,1,j)\big)^2 < \infty$.

3. $\lim\limits_{\alpha\to 1} M\big(\tau(\alpha,1,i)\big)^j = M\big(\tau(1,1,i)\big)^j,\ i\in\bar{H},\ j=1,2$.

4. $\lim\limits_{\alpha\to 1}(\mathrm{d})\,\xi(\alpha,1,i,1) = \xi(1,1,i,1),\ i\in\bar{H}$.

5. $\lim\limits_{\alpha\to 1} \sup\limits_{j\in\bar{H}} M\big(\xi(\alpha,1,i,1)\big)^2 < \infty$.

6. $\lim\limits_{\alpha\to 1} M\big(\xi(\alpha,1,i,1)\big)^j = M\big(\xi(1,1,i,1)\big)^j,\ i\in\bar{H},\ j=1,2$.

Remark 2. If one introduces the functionals

$$\hat{\xi}(\eta_0,\alpha,t) = \sum_{k=0}^{\nu(\alpha,t)} \gamma(\alpha,k) + \xi(\alpha,\nu(\alpha,t)+1,\eta_\alpha(\nu(\alpha,t)),t-\theta_{\nu(\alpha,t)}(\alpha)),$$

$$\hat{\tau}(\eta_0,\alpha,s) = \inf\{t : \hat{\xi}(\eta_0,\alpha,t) \ge s\}, \qquad s>0,$$

then for $\hat{\tau}(\eta_0,\alpha,s)$ the result of the theorem holds if, in addition to conditions (A), (B), (C), the condition

$$\lim_{u\to\infty} \lim_{\alpha\to 1} \sup_{x\in[0,\infty)} \sup_{j\in\bar{H}} P\{\tau(\alpha,1,j) > u+x \mid \tau(\alpha,1,j) > x\} = 0.$$

is fulfilled.

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References

1. A. V. **Skorokhod**, N. P. **Slobodyanyuk**, *Limit Theorems for Random Walks*, Kyiv, 1969.

Note: Figure translations are in progress. See original paper for figures.