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Abstract

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SOLUTION OF THE DIRICHLET PROBLEM FOR A NONLINEAR ELLIPTIC EQUATION BY INTRODUCING A CONTINUOUS PARAMETER

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In [1] the following was proved.

Theorem 1. Let $y = \Phi(x)$ be a continuous function mapping a Banach space X into a Banach space Y. Suppose that the equation

$$\Phi(x) = 0$$

has a unique solution x^* in the domain D:

$$\|x - x^*\| \le M.$$

Assume that in D there exist continuous Fréchet derivatives $\Phi'(x)$ and $\Phi''(x)$, as well as the inverse operator $\Phi'(x)^{-1}$, for which the inequality

$$\|\Phi'(x)^{-1}\| \le B$$

holds.

Then there exists a sphere $S:\ \|x - x^*\| \le \varepsilon$, belonging to the domain L, such that for any $x_0 \in S$ the differential equation

$$d\bar{x}(t)/dt = -\Phi'(x)^{-1}\Phi(x),$$

where t is a real parameter, with the initial condition

$$x(0) = x_0$$

has a unique solution $x(t)$ on the interval $0 \leq t < +\infty$, and

$$\lim_{t \to \infty} \|x(t) - x^*\|_X = 0.$$

In [1-4] Theorem 1 was used in solving boundary-value problems for nonlinear ordinary differential equations of the second and n-th orders, in solving nonlinear integral equations, and also in solving an inverse problem in scattering theory. In the present paper the method of stabilization with respect to a parameter is applied to the solution of a boundary-value problem for a nonlinear elliptic differential equation of second order. The formulation of the problem is given, a theorem on the introduction of a continuous parameter is formulated, and a scheme of an approximate method for solving the stated problem is presented.

I. Consider the nonlinear differential equation

$$\Delta z(x, y) + f(x, y, z) = 0, \tag{1}$$

where Δ is the Laplace operator, and $f(x, y, z)$ is a function twice continuously differentiable in its arguments. We seek a solution of equation (1), given in a bounded domain G, satisfying on the boundary Γ the condition

$$z(x, y)\big|_\Gamma = 0. \tag{2}$$

We assume that the boundary Γ of the domain G is sufficiently smooth.

Consider the complete linear normed space $X(G, H)$ of functions twice continuously differentiable in the closed domain $\bar{G}$

$z(x, y)$, vanishing on the boundary and such that the second derivatives of $z(x, y)$ satisfy, in $\overline{G}$, the Hölder condition with exponent λ $(0 < \lambda < 1)$. The norm of $z(x, y)$ in the space $X(G, H)$ is defined as follows:

$$\|z(x, y)\|_X = \sum_{l=0}^{2} \sum_{m=0}^{l} \max_{\overline{G}} \left| \frac{\partial^l z(x, y)}{\partial x^m \partial y^{l-m}} \right| + H_{z''_{x^2}} + H_{z''_{xy}} + H_{z''_{y^2}}, \tag{3}$$

where by $H_{z''_{x^2}}, H_{z''_{xy}}, H_{z''_{y^2}}$ we denote the lower bounds of the Hölder constants with exponent λ, respectively, for the functions $\partial^2 z/\partial x^2$, $\partial^2 z/\partial x \partial y$, $\partial^2 z/\partial y^2$ in the domain $\overline{G}$. Introduce the Banach space $Y(G, H)$ of functions $w(x, y)$ continuous in $\overline{G}$ and satisfying the Hölder condition with exponent λ. We define the norm in $Y(G, H)$ by the relation

$$\|w(x, y)\|_Y = \max_{\overline{G}} |w(x, y)| + H_w, \tag{4}$$

where H_w is the lower bound of the Hölder constants for the function $w(x, y)$ in the domain $\overline{G}$.

Theorem 2. Let, in the domain D

$$\|z - z^*\|_X \leq M \tag{5}$$

of the space $X(G, H)$, the boundary-value problem (1)–(2) have, and moreover uniquely, a solution $z^*(x, y)$. Suppose that for any $z(x, y) \in D$ and any $w(x, y) \in Y(G, H)$ the boundary-value problem

$$\Delta v(x, y) + f'_z(x, y, z)v(x, y) = w(x, y), \tag{6}$$

$$v(x, y)|_\Gamma = 0 \tag{7}$$

has a unique solution $v(x, y) \in X(G, H)$.

Then there exists a sphere S

$$\|z - z^*\|_X \leq \varepsilon, \qquad \varepsilon < M, \tag{8}$$

such that for any $z_0 \in S$ in the cylinder $\Omega = G \times [0 \leq t < +\infty)$ there exists a unique solution of the system of equations

$$\partial^2 v/\partial x^2 + \partial^2 v/\partial y^2 + f'_z(x, y, z)v(x, y, t) = -[\Delta z + f(x, y, z)],$$

$$\partial z(x, y, t)/\partial t = v(x, y, t), \tag{9}$$

satisfying the conditions:

$$v(x, y, t)|_B = 0,$$

$$z(x, y, t)|_{t=0} = z_0(x, y), \qquad z_0(x, y)|_\Gamma = 0 \tag{10}$$

(B is the lateral surface of the cylinder Ω), and, moreover,

$$\lim_{t \to \infty} \|z(x, y, t) - z^*(x, y)\|_X = 0. \tag{11}$$

A detailed proof of Theorem 2 is given in the authors' paper ([5]); here we merely note that for the proof one must verify that all the conditions of Theorem 1, formulated for a Banach space, are fulfilled in the case of our particular problem. In doing so, the theorem on boundedness of the inverse operator is used essentially (see, for example, ([6]), p. 157), as well as the theorems on Hölder continuity of the solution of an elliptic equation and on uniform estimation of the quantity $H_{v''}$ for the solution of the Dirichlet problem (6)–(7) (see ([7,8])).

II. Scheme of the numerical solution of problem (9)–(10). Choose the step of motion with respect to the parameter t, denote it by τ. Divide the domain $\Omega_T = G \times [0 \le t \le T]$ into n parts by planes parallel to the plane XOY:

$$t_0 = 0, \quad t_1 = \tau, \quad t_2 = 2\tau, \dots, t_n = n\tau = T.$$

Having an initial value $z(x, y, 0) = z_0(x, y)$, $z_0(x, y) \in S$, and substituting it into the first equation of system (9), we obtain a linear elliptic equation with respect to $v(x, y, 0)$

$$\Delta v(x, y, 0) + f'_z(x, y, z_0(x, y))v(x, y, 0) = -[\Delta z_0(x, y) + f(x, y, z_0(x, y))] \quad (12)$$

with the boundary condition

$$v(x, y, 0)\big|_\Gamma = 0. \quad (13)$$

Solving the boundary-value problem (12)–(13) by any known method (for example, by the grid method), we obtain the value of the function $v(x, y, t)$ on the layer $t_0 = 0$.

We now replace the second equation of system (9) by the difference relation

$$[z(x, y, t_1) - z_0(x, y)]/\tau = v(x, y, 0),$$

whence we find the value of the function $z(x, y, t)$ on the layer $t = t_1$.

In general, if the function $z(x, y, t)$ is known on the layer $t = t_k$, then the function $v(x, y, t)$ on this layer is determined by solving the linear boundary-value problem with respect to $v(z, y, t_k)$ as a function of x and y:

$$\begin{aligned} \Delta v(x, y, t_k) + f'_z(x, y, z(x, y, t_k))v(x, y, t_k) = \\ = -[\Delta z(x, y, t_k) + f(x, y, z(x, y, t_k))], \end{aligned} \quad (14)$$

$$v(x, y, t_k)\big| = 0. \quad (15)$$

Having thereby found $v(x, y, t_k)$, we determine the function $z(x, y, t)$ on the next layer $t = t_{k+1}$:

$$z(x, y, t_{k+1}) = z(x, y, t_k) + \tau v(x, y, t_k) \tag{16}$$

and so on.

In conclusion, we note that, on the basis of Theorem 2 of ([1]), one can draw the following conclusion:

If it is assumed that the solution of the boundary-value problem (14)–(15) is carried out exactly, then as $\tau \to 0$ we obtain convergence of the approximate solution to the exact one.

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Note: Figure translations are in progress. See original paper for figures.