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Abstract

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ON HIGHER APPROXIMATIONS OF N. N. BOGOLYUBOV–N. M. KRYLOV FOR SOLUTIONS BOUNDED ON THE WHOLE AXIS

(Presented by Academician N. N. Bogolyubov, 27 V 1968)

In the present paper we study, on an infinite interval of time, the higher approximations of the averaging method of N. N. Bogolyubov–N. M. Krylov ($^{1-3}$). The order of closeness between the exact solution of a differential equation and its asymptotic approximation is estimated. In this, the following theorem of P. P. Zabreiko, Yu. S. Kolesov, and M. A. Krasnosel' skii is used essentially (here E is a finite-dimensional real space, $|x|$ is the norm of an element $x \in E$).

Theorem 1 (^{4}). *Let the operator $X(t, x, \varepsilon)$ $(-\infty < t < \infty,\ |x| \leq a,\ |\varepsilon| \leq \varepsilon_0)$, with values in E, be bounded, continuous in t, and uniformly with respect to t continuous in x and ε; let, at the point $x = 0$, the operator $X(t, x, \varepsilon)$ have the Fréchet derivative $A(t, \varepsilon) = X'_x(t, 0, \varepsilon)$, and suppose that for $-\infty < t < \infty,\ |x_1|,\ |x_2| \leq r \leq a,\ |\varepsilon| \leq \varepsilon_0$ the inequality*

$$|X(t, x_1, \varepsilon) - X(t, x_2, \varepsilon) - A(t, \varepsilon)(x_1 - x_2)| \leq \omega(r)|x_1 - x_2|,$$

holds, where $\omega(r) \to 0$ as $r \to 0$. Finally, let the equalities

$$\lim_{\varepsilon \to 0} \left| \varepsilon \int_t^{t+1/\varepsilon} X(s, 0, \varepsilon)\, ds \right| = 0, \tag{1}$$

$$\lim_{\varepsilon \to 0} \left| \varepsilon \int_t^{t+1/\varepsilon} [A(s, \varepsilon) - B]\, ds \right| = 0, \tag{2}$$

be satisfied uniformly with respect to t, where B is a matrix having no zero and no purely imaginary eigenvalues.

Then there exist $a_0, \varepsilon_0 > 0$ such that, for $0 < |\varepsilon| \leq \varepsilon_0$, the equation

$$dx/dt = \varepsilon X(t, x, \varepsilon)$$

has a unique solution $x_\varepsilon(t)$, lying in the ball $|x| \leq a$ for all t, and moreover

$$\lim_{\varepsilon\to 0}\sup_{-\infty<t<\infty}|x_\varepsilon(t)|=0.$$

1. Consider the ordinary differential equation

$$dx/dt=\varepsilon X_0(x)+\varepsilon^2X_1(x)+\dots+\varepsilon^kX_{k-1}(x)+\varepsilon^{k+1}X_k(t,x,\varepsilon),\tag{3}$$

where $X_i(x)$ $(i=0,\dots,k{-}1)$, $X_k(t,x,\varepsilon)$ are operators acting in E. Suppose that there exists an operator $\bar{X}_k(x)$ such that, for $|x|\le a$, uniformly with respect to t, the equality

$$\lim_{\varepsilon\to 0}\left|\varepsilon\int_t^{t+1/\varepsilon}[X_k(s,x,\varepsilon)-\bar{X}_k(x)]\,ds\right|=0.$$

We shall assume that the averaged equation

$$dx/dt=\varepsilon X_0(x)+\varepsilon^2X_1(x)+\dots+\varepsilon^{kX}_{k-1}(x)+\varepsilon^{k+1}X_k(x)\tag{4}$$

has a stationary solution $x=u_\varepsilon$, i.e.

$$X_0(u_\varepsilon)+\varepsilon X_1(u_\varepsilon)+\dots+\varepsilon^{kX}_k(u_\varepsilon)=0.\tag{5}$$

Theorem 2. *Let $X_i(x)$ $(i=0,\dots,k-1;\ X_0(0)=0)$ be continuous bounded operators with values in E; let, for $|x|\le a$, the operators $X_i(x)$ have continuous Fréchet derivatives $B_i(x)$. Let the operator $X_k(t,x,\varepsilon)$ be bounded, continuous in t, uniformly with respect to t continuous in x and ε; let, for $|x|\le a$, the operator $X_k(t,x,\varepsilon)$ have a bounded derivative, continuous in t and uniformly with respect to t continuous in x, in the Fréchet sense*

$$B(t,x,\varepsilon)=[X_k(t,x,\varepsilon)]'_x.$$

Let the equalities

$$\lim_{\varepsilon\to 0}\left|\varepsilon\int_t^{t+1/\varepsilon}[X_k(s,0,\varepsilon)-X_k(0)]\,ds\right|=0,$$

$$\lim_{\varepsilon\to 0}\left|\varepsilon^{k+1}\int_t^{t+1/\varepsilon}B(s,x,\varepsilon)\,ds\right|=0$$

hold uniformly with respect to t.

Finally, let the matrix $B_0(0)$ have no zero or purely imaginary eigenvalues.

Then there exist $a_0, \varepsilon_0 > 0$ such that, for $0 < |\varepsilon| \le \varepsilon_0$, equation (3) has a unique solution $x_\varepsilon(t) = u_\varepsilon + \varepsilon^k y_\varepsilon(t)$, where u_ε is the stationary solution of equation (4), and

$$\lim_{\varepsilon\to 0} \sup_{-\infty<t<\infty} |y_\varepsilon(t)| = 0.$$

For the proof of the theorem, let us first note that, by the conditions of the theorem, for sufficiently small x and ε, equation (5) has a unique solution u_ε. Make in equation (3) the change of variable

$$x = u_\varepsilon + \varepsilon^k y.$$

As a result we obtain the new equation

$$dy/dt = \varepsilon V(t, y, \varepsilon),$$

where

$$V(t, y, \varepsilon) = X_0(u_\varepsilon + \varepsilon^k y)/\varepsilon^k + X_1(u_\varepsilon + \varepsilon^k y)/\varepsilon^{k-1} + \cdots$$

$$\cdots + X_{k-1}(u_\varepsilon + \varepsilon^k y)/\varepsilon + X_k(t, u_\varepsilon + \varepsilon^k y, \varepsilon).$$

It remains to note that the operator $V(t, y, \varepsilon)$ satisfies the conditions of Theorem 1.

Let us verify, for example, condition (1):

$$\lim_{\varepsilon\to 0} \left| \varepsilon \int_t^{t+1/\varepsilon} V(s, 0, \varepsilon)\, ds \right| =$$

$$= \lim_{\varepsilon\to 0} \left| \varepsilon \int_t^{t+1/\varepsilon} \left[\frac{X_1(u_\varepsilon)}{\varepsilon^k} + \frac{X_1(u_\varepsilon)}{\varepsilon^{k-1}} + \cdots + \frac{X_{k+1}(u_\varepsilon)}{\varepsilon} + X_k(s, u_\varepsilon, \varepsilon) \right] ds \right| =$$

$$= \lim_{\varepsilon\to 0} \left| \varepsilon \int_t^{t+1/\varepsilon} [X_k(s, u_\varepsilon, \varepsilon) - X_k(u_\varepsilon)]\, ds \right| = 0.$$

The remaining conditions are checked analogously.

2. Let $X_i(t, x)$ $(-\infty < t < \infty,\ |x| \le a,\ i = 0, \dots, k-1)$ be operators continuous in t and smooth in x; let $X_k(t, x, \varepsilon)$ $(-\infty < t < \infty,\ |x| \le a,\ |\varepsilon| \le \varepsilon_0)$ be an operator continuous in t and smooth in x and ε.

Consider in E the ordinary differential equation

$$dx/dt = \varepsilon X_0(t,x) + \ldots + \varepsilon^k X_{k-1}(t,x) + \varepsilon^{k+1} X_k(t,x,\varepsilon). \tag{6}$$

Following (1-3), in equation (6) we make a substitution of the form

$$x = y + \varepsilon U_1(t,y) + \ldots + \varepsilon^k U_k(t,y)$$

so that the new equation becomes autonomous up to terms of order k in ε:

$$dy/dt = \varepsilon Y_0(y) + \ldots + \varepsilon^k Y_{k-1}(y) + \varepsilon^{k+1} Y_k(t,y,\varepsilon). \tag{7}$$

For this, the operators $Y_i(y), U_{i+1}(t,y)$ $(i = 0, \ldots, k-1)$ are successively determined from the equalities

$$Y_i(y) = \lim_{T\to\infty} \frac{1}{T} \int_0^T F_i(\tau,y)\,d\tau,$$

$$U_{i+1}(t,y) = \int_0^t [F_i(\tau,y) - Y_i(y)]\,d\tau;$$

the operators $F_i(t,y)$ $(i = 0, \ldots, k-1)$ and $Y_k(t,y,\varepsilon)$ are written out according to the recurrent formulas given, for example, in ([5]).

As shown in ([5]), the substitution described above is legitimate and, moreover, the operators $Y_i(y)$ $(i = 0, \ldots, k-1)$ are continuously differentiable $k-i$ times if the operators $X_i(t,x)$ are continuously differentiable $k-i$ times and if, for all $T \to \infty$, the operators

$$\frac{1}{T} \int_0^T F_i(\tau,y)\,d\tau$$

converge to the operator $Y_i(y)$, together with derivatives up to order $k-i$, uniformly for all $\|y\| \le a$.

We shall assume that the equalities

$$\lim_{T\to\infty} \sup_{-\infty<t<\infty} \left| \frac{1}{T} \int_t^{t+T} F_i(\tau,y)\,d\tau - Y_i(y) \right| = 0$$

hold and that, as $\varepsilon \to 0$, the operator $Y_k(t,y,\varepsilon)$ tends "in the averaged sense" to some $Y_k(y)$, i.e.

$$\lim_{\varepsilon\to 0}\sup_{-\infty<t<\infty}\left|\varepsilon\int_t^{t+1/\varepsilon}[Y_k(s,y,\varepsilon)-Y_k(y)]\,ds\right|=0.$$

Let $y_\varepsilon(t)$ be a solution of equation (7), and let $\bar{y}_\varepsilon$ be a stationary solution of the equation

$$dy/dt=\varepsilon Y_0(y)+\varepsilon^2 Y_1(y)+\ldots+\varepsilon^{kY}_{k-1}(y)+\varepsilon^{k+1}Y_k(y).$$

Put

$$x_\varepsilon(t)=y_\varepsilon(t)+\varepsilon U_1[t,y_\varepsilon(t)]+\ldots+\varepsilon^k U_k[t,y_\varepsilon(t)],$$

$$\bar{x}_\varepsilon(t)=\bar{y}_\varepsilon+\varepsilon U_1(t,\bar{y}_\varepsilon)+\ldots+\varepsilon^k U_k(t,\bar{y}_\varepsilon).$$

The function $\bar{x}_\varepsilon(t)$ is called an asymptotic approximation (in the sense of N. N. Bogolyubov-N. M. Krylov) of order $k+1$ to the function $x_\varepsilon(t)$, the exact solution of equation (6).

We shall say that the conditions $(\mathfrak{D}_k)$ are satisfied for equation (6) if:

a) the operators $X_i(t,x)$ $(i=0,\ldots,k-1)$, together with their derivatives up to order $k-i$, are continuous and bounded;

b) the operator $Y_k(t,x,\varepsilon)$ is bounded, continuous in t, and uniformly with respect to t continuous in x and ε; as $\varepsilon\to 0$ it "in the averaged sense" tends to

to the continuous operator $Y_k(x)$:

$$\lim_{\varepsilon\to 0}\sup_{-\infty<t<\infty}\left|\varepsilon\int_t^{t+1/\varepsilon}[Y_k(s,x,\varepsilon)-Y_k(x)]\,ds\right|=0;$$

c) the operator $Y_k(t,x,\varepsilon)$, for $|x|\le a$, has a Fréchet derivative $B(t,x,\varepsilon)=[Y_k(t,x,\varepsilon)]'_x$ that is bounded, continuous in t, and uniformly with respect to t continuous in x and ε, and, moreover,

$$\lim_{\varepsilon\to 0}\left|\varepsilon^{k+1}\int_t^{t+1/\varepsilon}B(s,x,\varepsilon)\,ds\right|=0.$$

d) the operators $U_i(t,x)$ $(i=1,\ldots,k)$ are bounded.

Verification of these restrictions is especially simple if the operators X_i are periodic or almost periodic in t.

Theorem 3. *Suppose that for the differential equation (6) the conditions* $(\mathfrak{D}_k)$ *are satisfied. Suppose* $Y_0(0) = 0$*, and the matrix* $B_0(0)$ *has no zero or purely imaginary eigenvalues. Then*

$$\lim_{\varepsilon\to 0} \sup_{-\infty<t<\infty} \frac{|x_\varepsilon(t) - \bar{x}_\varepsilon(t)|}{\varepsilon^k} = 0.$$

3. Let us note that, in the case when the right-hand sides of equation (6) are, uniformly with respect to x, almost periodic in t (ω-periodic in t), the corresponding solution also turns out to be almost periodic (ω-periodic) in t. The results of the paper are naturally carried over to equations in Banach spaces; in this case one must require of the operator $B_0(0)$ that its spectrum not intersect the imaginary axis.

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