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Abstract

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MATHEMATICS

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SOME APPLICATIONS OF THE LOCAL CRITERION FOR THE EXISTENCE OF WAVE OPERATORS

(Presented by Academician V. I. Smirnov on 12 XII 1967)

In note ([1]) the author proposed a fairly general "local" criterion for the existence of wave operators. Here we indicate some consequences of this criterion, giving convenient sufficient conditions for the existence of wave operators "as a whole," and also give applications to a number of scattering problems for differential operators.

1. Let H^0, H be self-adjoint operators in a Hilbert space $\mathfrak{H}$. Below, $\mathfrak{D}(H), E(\cdot), P$ denote respectively the domain, the spectral measure, and the projector onto the absolutely continuous subspace for H. The notations $\mathfrak{D}(H^0), E^0(\cdot), P^0$ have the analogous meaning for H^0. If the wave operators

$$W_\pm(H, H^0) = \operatorname*{s-lim}_{t\to\pm\infty} \exp(itH)\exp(-itH^0)P^0, \tag{1}$$

exist, then the scattering operator $S = W_+^* W_-$ commutes with H^0. Let $P^0\mathfrak{H}$ be decomposed into a direct integral, where the operator H^0P^0 becomes the operator of multiplication by the independent variable λ. In this representation the operator S corresponds to "multiplication" by the operator-function S_λ.

If the operators $W_\pm(H, H^0)$ exist and, for almost all λ, S_λ differs from the identity operator by a nuclear operator*, then the ordered pair H^0, H will be called **subnormal**. If, in addition, the operators $W_\pm(H^0, H)$ exist, then the pair H^0, H will be called **normal**. We note that for a normal pair the operators H^0P^0 and HP are unitarily equivalent.

We shall agree to say that the operator H^0 is **subordinate** to the operator H if, for some continuous functions φ, φ^0 such that $|\varphi^0(\lambda)| \to \infty$ and $|\varphi(\lambda)| \to \infty$ as $|\lambda| \to \infty$, one has

$$\mathfrak{D}(\varphi(H)) \subseteq \mathfrak{D}(\varphi^0(H^0)).$$

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Theorem 1. Let the orthoprojectors M_k^0 commute with H^0 and $\sup_k M_k^0 = I$. The pair H^0, H is subnormal if, for any $k = 1, 2, \ldots$, the operators HM_k^0 are defined on $\mathfrak{D}(H^0)$ and

$$HM_k^0 - H^0 M_k^0 \in \mathfrak{S}_1.$$

Theorem 2. The pair H^0, H is normal if the operator H^0 is subordinate to H and, for any finite interval δ,

$$HE^0(\delta) - H^0 E^0(\delta) \in \mathfrak{S}_1.$$

Theorem 3. The pair H^0, H is normal if the operators H^0 and H are subordinate to each other and, for any finite interval δ,

$$HE(\delta)E^0(\delta) - E(\delta)H^0 E^0(\delta) \in \mathfrak{S}_1.$$

From a technical point of view, a convenient means of verifying the conditions of Theorems 1–3 is provided by the criteria for nuclearity of integral operators obtained in ([2],[3]).

* The class of nuclear operators is denoted below by γ_1, that of completely continuous operators by γ_∞.

2. Below, $a(x)$ (with various indices) will always denote bounded functions of the point $x \in R^m$ satisfying, for some $\beta > m/4$, the condition $a(x)(1 + |x|^2)^\beta \in L_2(R^m)$. As usual, $D_k = i\partial/\partial x_k$, $D^\omega = D_1^{\omega_1} \cdots D_m^{\omega_m}$.

Let $\mathfrak{H} = L_2(R^m)$ and $H^0 = (-\Delta)^l$. The formally self-adjoint expression

$$(-\Delta)^l + \sum_{|\omega| \leq s} a_\omega(x) D^\omega$$

naturally generates a symmetric operator H' on $\mathfrak{D}(H^{0r})$, if $2lr \geq s$. Suppose that H' has self-adjoint extensions, and let H be any one of them.

Theorem 4. *The pair H^0, H is subnormal. The pair H^0, H is normal if*

$$(H^0 u, u) \leq c[(H'u, u) + (u, u)] \qquad (u \in \mathfrak{D}(H^{0r}))$$

and H is the Friedrichs extension of the operator H'.

Note that the pair H^0, H is certainly normal if $s < 2l$. This strengthens the results of I. V. Stankevich [4], who imposed considerably more stringent requirements on $a_\omega(x)$. It is of interest, however, that nonnegative perturbation terms of arbitrarily high order may be introduced into H^0.

3. Perturbations of order not higher than $2l$ may be introduced into the operator $H^0 = (-\Delta)^l$ by means of bilinear forms, which requires no smoothness of the coefficients. Let

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$$V[u,u]=\int\sum_{|\sigma|\le l,\ |\rho|\le l}a_{\sigma\rho}(x)(D^{\sigma}u\,\overline{D^{\rho}u}+D^{\rho}u\,\overline{D^{\sigma}u})\,dx \tag{2}$$

and let the terms in (2) corresponding to $|\sigma|=|\rho|=l$ form a nonnegative form. The quadratic form

$$\|\sqrt{H^0}u\|^2+V[u,u]\qquad(u\in\mathfrak{D}(\sqrt{H^0}))$$

is semibounded and closed. It corresponds to a certain self-adjoint operator H.

Theorem 5. *The pair H^0, H is normal.*

We note that, under the assumptions of items 2 and 3, the requirement of boundedness of the coefficients could be weakened, and in some cases removed altogether. Furthermore, Theorems 4 and 5 remain valid if one sets $H^0=P(D)$, where $P(\xi)$ is an arbitrary real elliptic polynomial.

4. The ellipticity requirement for the unperturbed operator is not necessary for the applicability of Theorems 1-3. Let $P(\xi)$ be a real polynomial. We shall call a polynomial $Q(\xi)$ *admissible* for $P(\xi)$ if, for every $\alpha>0$,

$$\overline{Q(\xi)}Q(\xi)\le\alpha P^2(\xi)+C(\alpha). \tag{3}$$

It is clear that in this case the degree of Q must be less than the degree of P.

Along with the operator $H^0=P(D)$, consider the self-adjoint operator H generated by the formally self-adjoint expression

$$P(D)-\sum_{\tau}a_{\tau}(x)Q_{\tau}(D). \tag{4}$$

Theorem 6. *The pair H^0, H is subnormal if the polynomials $Q_\tau(\xi)$ are admissible for $P^r(\xi)$ for some $r\ge 1$. The pair H^0, H is normal if $r=1$ and*

$$\lim_{|\xi|\to\infty}|P(\xi)|=\infty. \tag{5}$$

Condition (5) singles out a class of operators $P(D)$, called in ([5]) strongly Carleman (s. c.) operators. This class contains hypoelliptic operators. For $m>1$, s. c. operators are semibounded. On this basis one could introduce, as in Theorem 4, perturbations of higher orders that give a normal pair. One can also indicate an analogue of Theorem 5.

Let us note that condition (5) also admits a weakening. Thus, for example, by Theorem 3, a normal pair in $L_2(R^2)$ is formed by the operators $P(D)$, $P(D)+a(x)$ for $P(\xi)=\xi_1^4-\xi_2^4$ (cf. ([6]), where the scattering problem is considered by Friedrichs' method for $P(\xi)=\xi_1^2-\xi_2^2$).

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5. The results of Sec. 4 carry over to the case of matrix differential expressions. Let $P(\xi)$ be a polynomial Hermitian $(n \times n)$-matrix. Definition (3) of admissibility of the matrix $Q(\xi)$ for $P(\xi)$ retains its meaning if $Q(\xi)$ is understood to be the Hermitian adjoint matrix. We shall call the matrix $P(\xi)$ strongly Carleman (s. c.) if its eigenvalue $p(\xi)$ least in absolute value satisfies condition (5). Of course, an elliptic matrix is always an s. c. matrix.

Let $H^0 = P(D)$, and let the self-adjoint operator H be generated by a formally self-adjoint expression of the form (4), where $a_\tau(x)$ and $Q_\tau(\xi)$ are certain $(n \times n)$-matrices.

Theorem 7. *The pair H^0, H is subnormal if all $Q_\tau(\xi)$ in (3) are admissible for $P_\tau(\xi)$ for some $r \geqslant 1$. The pair H^0, H is normal if $r = 1$ and $P(\xi)$ is an s. c. matrix.*

Hence, in particular, it follows that under perturbation of a Hermitian matrix $P(D)$ by an operator $a(x)$, there always exist wave operators (1).

As examples of application of the results of Secs. 2–5 one may cite quantum scattering problems for a nonrelativistic particle in a potential field, in an electromagnetic field, with spin-orbit interaction taken into account, etc., as well as the problem of scattering by an external field of a relativistic electron (the Dirac system). In all these cases a normal pair arises. The indicated problems have been studied in a number of works by individual methods (see, for example, ([7],[8])). However, here too the information about the spectral properties of the scattering matrix S_λ is new.

6. In a number of questions, scattering problems naturally arise for pairs of operators acting in different spaces. One can obtain an analogue of the local criterion ([1]) for the existence of wave operators in this case. Here we shall give only the simplest consequences of the corresponding local criterion.*

Let the self-adjoint operators H^0, H act in the spaces $\mathfrak{H}^0, \mathfrak{H}$, respectively, and let the bounded and boundedly invertible operator J map $\mathfrak{H}^0$ into $\mathfrak{H}$. Following T. Kato ([9]), define the wave operators

$$W_\pm(H, H^0; J) = s\text{-}\lim_{t \to \pm\infty} \exp(itH) J \exp(-itH^0) P^0.$$

We shall call the triple H^0, H, J **subnormal** if there exist operators (6) isometric on $P^0\mathfrak{H}^0$, and the scattering matrix S_λ differs from the identity operator by a nuclear one. We shall call the triple H^0, H, J **normal** if, in addition, there exist operators $W_\pm(H^0, H; J^*)$ isometric on $P\mathfrak{H}$. In the latter case the absolutely continuous parts of the operators H^0 and H are unitarily equivalent.

Theorem 8. *Let the orthoprojectors M_k^0 commute with H^0 and $\sup_k M_k^0 \neq I^0$. The triple H^0, H, J is subnormal if, for any $k = 1, 2, \ldots$, the operators HJM_k^0*

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are defined on $\mathfrak{D}(H^0)$,

$$HJM_k^0 - JH^0M_k^0 \in \mathfrak{S}_1, \qquad (J^*J - I^0)M_k^0 \in \mathfrak{S}_\infty.$$

* The results of Sec. 6 were obtained jointly with A. L. Belopol' skii.

Theorem 9. The triple H^0, H, J is normal if $J\mathfrak{D}(H^0) = \mathfrak{D}(H)$ and, for every finite interval δ,

$$HJE^0(\delta) - JH^0E^0(\delta) \in \mathfrak{S}_1, \qquad (J^*J - I^0)E^0(\delta) \in \mathfrak{S}_\infty.$$

7. Let b_0 be a constant positive definite $(n \times n)$-matrix; let $b(x)$ be another matrix, with $0 < c_1 \leq b(x) \leq c_2$, and let $a(x) = b(x) - b_0$ satisfy the condition introduced in Sec. 2. On the set of n-component vector functions $u(x)$, in addition to the usual scalar product in $L_2(R^m)$, introduce the scalar products $[u, v]_0 = (b_0, u, v)$, $[u, v] = (bu, v)$. These define two new Hilbert spaces $\mathfrak{H}^0, \mathfrak{H}$, which are metrically equivalent to $L_2(R^m)$. Let $P(\xi)$ be a polynomial Hermitian matrix, $H^0 = b_0^{-1}P(D)$, $H = b^{-1}P(D)$, and let J be the identity operator. The operators H^0 and H are self-adjoint in $\mathfrak{H}^0$ and $\mathfrak{H}$, respectively.

Theorem 10. The triple H^0, H, J is subnormal. If $b_0^{-1}P(\xi)$ is a s.c. matrix, then the triple H^0, H, J is normal.

Let $P(\xi)$ be a linear homogeneous function of ξ. For one special class of such matrices, Wilcox ([10]) (see also Kato ([9])) proved (under a somewhat weaker decrease of $a(x)$) the existence of the isometric operators (6). The more difficult question of the existence of the operators $W_\pm(H^0, H; J^*)$ remained open. The first assertion of Theorem 10 shows that the existence of the operators (6) is not, in general, connected with any special properties of the matrix $P(\xi)$. The situation is otherwise with the second assertion of Theorem 10. The requirement that $b_0^{-1}P(\xi)$ be strongly Carleman means Petrovskii hyperbolicity for the corresponding system with time $b_0u_t = P(D)u$. This condition is not satisfied, for example, for Maxwell' s system. The latter can, however, be considered separately, at least for the case of an isotropic medium.

Theorem 11. Let $u = \{u_1, u_2\}$, where u_1, u_2 are three-component functions in R^3, $P(D)u = \{\operatorname{rot} u_2, -\operatorname{rot} u_1\}$. Let $b_0, b(x)$ be scalars and let $\operatorname{grad} b(x)$ be locally bounded. Then the operators H^0, H, J defined above form a normal triple.

We note that Theorem 11 is not a consequence of Theorem 9, but is proved in a similar way on the basis of the general local criterion for the existence of wave operators.

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