ON THE STRUCTURE OF A NEIGHBORHOOD OF A SINGULAR POINT OF A TWO-DIMENSIONAL AUTONOMOUS SYSTEM

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Abstract

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ON THE STRUCTURE OF A NEIGHBORHOOD OF A SINGULAR POINT OF A TWO-DIMENSIONAL AUTONOMOUS SYSTEM

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Consider the autonomous system

$$\dot{x} = X_m(x,y) + \varphi(x,y) \equiv X(x,y), \qquad \dot{y} = Y_m(x,y) + \psi(x,y) \equiv Y(x,y), \quad (1)$$

where X_m, Y_m are homogeneous polynomials of degree $m \geq 1$, $X_m^2 + Y_m^2 \neq 0$, $\varphi, \psi = o(r^m)$, $r = \sqrt{x^2+y^2}$, and suppose that the origin O is its isolated singular point. We shall call system (1) a $C_{n,m}$-system if, in some neighborhood of the point O, $X, Y \in C_n$, $n \geq 0$ ($C_0 \equiv C$), and an A_m-system if X and Y are holomorphic in a neighborhood of this point. In the author' s note ([1]) it was shown that only a finite number of elliptic regions can adjoin the singular point O of a $C_{m,m}$-system. Therefore such a system has at the point O a local topological structure ([2]). We divide the set of topological structures of the singular point O arising for all possible $C_{m,m}$-systems (A_m-systems) into classes, assigning to one and the same class all and only those structures for which both the number e of elliptic regions and the number h of hyperbolic regions are the same. In the present note the following problem is solved: to find all pairs of numbers (e,h) that correspond to classes generated by A_m-systems for fixed m, and, in each class, to establish the topological structure of an unstable singular point ([3]) with the minimal number of parabolic regions for the given class. The analogous problem is also considered for $C_{m,m}$-systems.

Consider the curve Γ defined by the equation $p_m(x,y) + \delta(x,y) = 0$, where $p_m \neq 0$ is a homogeneous polynomial of degree $m \geq 1$, $\delta = o(r^m)$, $\delta(0,0) = 0$, and $\delta \in C$ in some neighborhood of the origin. Let U be an arbitrary sufficiently small neighborhood of the point O, and let Γ_U be the component of this point in the set $\Gamma \cap U$. The components of the set $\Gamma_U \setminus \{O\}$ will be called O-branches of the curve Γ. Any connected subset of an O-branch for which the point O is a limit point will be called an essential part of this branch.

Lemma 1. *Every elliptic region of a $C_{0,m}$-system (1), and also every one of its hyperbolic regions whose closure contains no O-branch of the line $x = 0$ ($y = 0$), contains an essential part of an O-branch of the curve $X = 0$ ($Y = 0$).*

Taking into account the proof of the lemma from ([1]), we obtain

Corollary 1. *For $C_{m,m}$-systems the inequality holds*

$$e + h \leq 2m + 2. \tag{2}$$

From inequality (2) and Theorem 2 of ([4]) it follows:

Corollary 2. *For $C_{m,m}$-systems the Bendixson formula is valid,*

$$e - h = 2(j - 1),$$

where j is the index of the singular point O.

Corollary 3. *For every $C_{m,m}$-system: 1) the numbers e and h have the same parity, 2) the inequalities are satisfied*

$$2m - 2 \leq e - h \leq 2m - 2. \tag{3}$$

Lemma 2. Each curvilinear semitrajectory of the A_m-system that adjoins the singular point O with bounded polar angle ϑ, sufficiently close to this point, admits a representation $r = r(\vartheta)$, where the function $r(\vartheta)$ is single-valued, differentiable, and strictly monotone.

Corollary. In a sufficiently small neighborhood of the origin of coordinates, each elliptic (hyperbolic) domain of the A_m-system either has no common point with the ray $\vartheta = \vartheta_0$, or is cut by it into two connected parts.

We shall say that an elliptic (hyperbolic, parabolic) domain of the A_m-system adjoins the ray $\vartheta = \vartheta_0$ if the closure of this domain contains either some segment of the ray with one endpoint at the point O, or a semitrajectory tangent to this ray at the point O. If the domain adjoins a single ray, then it is called degenerate; otherwise, nondegenerate. Introduce the notation: $F = xY - yX$, $G = xX + yY$, $F_{m+1} = xY_m - yX_m$, $G_{m+1} = xX_m + yY_m$. We shall say that an elliptic (hyperbolic) domain belongs to the ray $\vartheta = \vartheta_0$ if at least one of the O-branches of the curve passing through it

$$G = 0 \tag{4}$$

is tangent to this ray at the point O. The rays of critical direction of Frommer (5) will be called critical rays.

Main lemma. If for the A_m-system $e \neq 0$, then the inequality

$$e + h \leq 2m. \tag{5}$$

holds.

Proof. If $F_{m+1} \equiv 0$, where $X_m = xf_{m-1}$, $Y_m = yf_{m-1}$, then the branching index of the curve (4) at the point O, and therefore also $e+h$, does not exceed $2m-2$. Therefore suppose that $F_{m+1} \neq 0$, and let the A_m-system (1) first possess a nondegenerate elliptic domain. By rotating the coordinate system through a suitable angle we shall ensure that not fewer than two essentially distinct O-branches of the curve $Y = 0$ pass through this domain, with $Y_m \neq 0$, so that the branching index of this curve at the point O does not exceed $2m$. Hence, by Lemma 1 and the evenness of the number $e+h$, we obtain (5).

Now let system (1) have no nondegenerate elliptic domains. Align the ray to which some degenerate elliptic domain belongs with the positive direction of the x-axis. By choosing a new y-axis we may assume that $X_m \neq 0$. We shall argue by contradiction: suppose that $e+h = 2m+2$. Then, by Lemma 1, the curve $X = 0$ will have $2m$ essentially distinct O-branches, and one of them at the point O is certainly tangent to the x-axis. We have (cf. (1))

$$X(x,y) = x^{\sigma}(y - \alpha_1(x)) \cdots (y - \alpha_{m-\sigma}(x))E(x,y),$$

where $\sigma = 0$ or 1, E is a unit divisor, the functions $\alpha_i(x)$ are holomorphic in a neighborhood of $x = 0$ $(i = 1, 2, \dots, m-\sigma)$, $\alpha_i(0) = 0$, and without loss of generality one may assume that $\alpha'_{m-\sigma}(0) = 0$. The regular analytic transformation $x \to x$, $y \to y + \alpha_{m-\sigma}(x)$ brings system (1) into

$$\begin{aligned} \dot{x} &= x^{\sigma} y (y - \beta_1(x)) \cdots (y - \beta_{m-\sigma-1}(x)) E(x, y + \alpha_{m-\sigma}(x)) \\ &\equiv y(g_{m-1} + \bar{\varphi}) \equiv \bar{X}, \\ \dot{y} &= Y(x, y + \alpha_{m-\sigma}(x)) - \alpha'_{m-\sigma}(x)\bar{X}(x,y) \equiv \bar{Y}_m + \bar{\psi} \equiv \bar{Y}, \end{aligned} \tag{6}$$

where $\beta_i = \alpha_i - \alpha_{m-\sigma}$ $(i = 1, 2, \dots, m-\sigma-1)$, $g_{m-1} \neq 0$ and $\bar{Y}_m$ are homogeneous polynomials of dimensions $m-1$ and m, $\bar{\varphi} = o(r^{m-1})$, $\bar{\psi} = o(r^m)$. With the aid of Lemma 1 it is easy to see that the curve $\bar{Y} = 0$ must have not fewer than $2m+2$ O-branches. Consequently, $\bar{Y}_m \equiv 0$, and the equation corresponding to system (6) takes the form

$$dy/dx = \bar{\psi}(x,y) \,/\, y\,[g_{m-1}(x,y) + \bar{\varphi}(x,y)].$$

Since for the last equation $F_{m+1} = y^2 g_{m-1}$, $G_{m+1} = xyg_{m-1}$, each elliptic and hyperbolic region belongs either to one of the critical rays, or to the ray $\vartheta = \pi/2$ or $\vartheta = 3\pi/2$. From the equality $e+h = 2m+2$ it follows that each such ray will be assigned as many regions as is the maximal degree in which the corresponding factor $y - kx$ or x occurs in G_{m+1}. Let us consider successively the directions whose angular coefficients are 0, ∞, and $k_0 \neq 0, \infty$, assuming that each of them is critical, except, perhaps, $k = \infty$, of multiplicity n (for $k = \infty$ we take $n \geq 0$).

1) $k = 0$. The number of all elliptic and hyperbolic regions belonging to the direction $k = 0$, i.e., to the positive or negative semiaxis of the x-axis, is $2n-2$. It is easy to see that through the elliptic region cut by the positive semiaxis of the x-axis there pass two O-branches of the curve

$$F = 0 \tag{7}$$

and one O-branch of the curve (4), while through each of the remaining $n-2$ elliptic and hyperbolic regions belonging to this semiaxis there passes one O-branch of both curve (4) and curve (7). Each of the two extreme regions adjacent to this semiaxis will be either a nondegenerate hyperbolic region belonging to the positive semiaxis of the x-axis, or a parabolic one. It is not hard to show that through each of them there must pass at least one O-branch of curve (7), tangent to the neighboring critical ray. We shall say that the positive semiaxis of the x-axis lays claim to at least two "foreign"O-branches of curve (7), or, more briefly: it has no fewer than two claims. The negative semiaxis of the x-axis will cut either an elliptic (parabolic) region—then an analogous situation arises for it —or a hyperbolic one. In the latter case, no more than two parabolic regions or hyperbolic regions not belonging to it may adjoin the negative semiaxis, through each of which there passes an O-branch of curve (7), tangent to this semiaxis. We shall say that the negative semiaxis of the x-axis has no more than two O-branches of curve (7) vacant for neighboring critical rays, or, more briefly: no more than 2 vacancies.

2) $k = k_0 \neq 0, \infty$. To the critical direction k_0 there belong $2n$ elliptic and hyperbolic regions, while $2n + 2$ such regions may adjoin it. As before, we conclude that the direction k_0 will have no fewer than 2 claims and no more than 2 vacancies.

3) $k = \infty$. The number of all elliptic and hyperbolic regions, both belonging to this direction and adjoining it, is $2n + 2$. We have 4 claims and not a single vacancy.

Considering all critical directions and the direction $k = \infty$, we conclude that the number of all claims is at least 4 greater than the number of all vacancies, which is contradictory. The lemma is proved.

Theorem. *All classes generated by A_m-systems are determined by the following table of $(m+1)^2$ ordered pairs of numbers (e, h):*

$(0,0)$	$(0,2)$	$(0,4)$	...	$(0,2m-2)$	$(0,2m)$	$(0,2m+2)$
	$(1,1)$	$(1,3)$	...	$(1,2m-3)$	$(1,2m-1)$	
$(2,0)$	$(2,2)$	...	$(2,2m-4)$	$(2,2m-2)$		
...	...	...	...	...		
	$(2m-3,1)$	$(2m-3,3)$				
$(2m-2,0)$	$(2m-2,2)$					
	$(2m-1,1)$					

(A)

The topological structure of an unstable singular point having the minimal, for the given class, number of parabolic regions is as follows: around the singular point there are arranged in succession h hyperbolic regions, separated from one another by separatrices; then, alternating with one another, follow p parabolic and e elliptic regions, so that $p = e + 1$ if $eh \neq 0$ or $e = h = 0$, and $p = e$ in the remaining cases. For classes,

for the corresponding $2m$ pairs in the first two columns of table (A), this structure is the only possible one.

Proof. From inequality (2), the main lemma, and Corollary 3 it follows that only the cases listed in table (A) are not excluded. The realizability of each of these cases is shown by constructing the corresponding examples. At the same time, the remaining assertions of the theorem are easily verified. Thus, the differential equation

$$\frac{dy}{dx} = \frac{x(x^2+2y^2)^k\, y\,(y+2x^2)\ldots[y+2(m-2k-i-1)x^2]}{(x^2+y^2)^k\,(y+x^2)(y+3x^2)\ldots[y+(2m-4k-2i-1)x^2]}\times$$

$$\times\frac{[y-2(m-2k-i)x^2]\ldots[y-2(m-2k-1)x^2]}{[y-(2m-4k-2i+1)x^2]\ldots[y-(2m-4k-1)x^2]} \equiv \frac{Y}{X}$$

$$(k = 0, 1, \ldots, [(m-1)/2]; \quad i = 0, 1, \ldots, m-2k)$$

for $k = 0$ and i varying from zero to m realizes the cases $(2m-1,1)$, $(2m-3,3), \ldots, (3,2m-3), (1,2m-1), (0,2m+2)$; for $k = 1$ and the corresponding variation of i, the cases $(2m-5,1)$, $(2m-7,3), \ldots, (3,2m-7), (1,2m-5), (0,2m-2)$; ...; for $k = [(m-1)/2]$, when m is even $(i = 0,1,2)$, the cases $(3,1), (1,3), (0,6)$, and when m is odd $(i = 0,1)$, the cases $(1,1), (0,4)$. Assigning to k and i suitable values, drawing the O-branches of the curves $X = 0$, $Y = 0$, and taking into account the sign of the derivative dy/dx between them, we obtain the corresponding topological structure with the minimal number of parabolic regions.

Corollary. For $C_{m,m}$-systems, apart from the cases of table (A) that are certainly realizable, the possibility of only the following pairs is not excluded: $(1,2m+1)$, $(2,2m), \ldots, (2m-1,3), (2m,2)$.

This follows from the theorem, estimate (2), and Corollary 3 (inequalities (3) exclude the pairs $(2m, 0)$, $(2m+1, 1)$, and $(2m+2, 0)$).

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Note: Figure translations are in progress. See original paper for figures.