# ON THE STRUCTURE OF OPTIMAL STRATEGIES

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## Abstract

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# ON THE STRUCTURE OF OPTIMAL STRATEGIES

# IN SOME MATRIX GAMES

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In this note, for matrix games, the existence is established of an optimal strategy representable in a special form, as well as of a strategy equivalent to an arbitrary mixed strategy and representable in the same form. The results obtained are applied to the study of the structure of such strategies in games that are sums of matrix games, in Blotto games, and in some other games.

Consider a matrix game $\Gamma$ with matrix $A = \|a_{ij}\|$ $(i = 1, 2, \dots, m;\ j = 1, 2, \dots, n)$. Let $A_j$ be the $j$-th column of the matrix $A$.

Denote by $X$ the set of mixed strategies, by $X^0$ the set of optimal strategies, and by $X^\varepsilon$ the set of $\varepsilon$-optimal strategies of the first player in the game $\Gamma$.

Let $I_1$ be the set of indices $i$ for which there exists an optimal strategy $x^i$ such that $x_i^i > 0$.

Strategies $x^1$ and $x^2$ of the first player will be called equivalent if $x^1 A = x^2 A$, and $\varepsilon$-equivalent if

$$\max_j |(x^1 - x^2) A_j| \leqslant \varepsilon.$$

Let $I_x$ be the set of indices $i$ for which there exists a strategy $x^i$, equivalent to $x$, such that $x_i^i > 0$.

**Theorem 1.** 1°. There exist $x \in X^0$, a number $a$, and a vector $y = (y_1, y_2, \dots, y_n)$, such that for $i \in I_1$

$$x_i = a \exp\left(\sum_{j=1}^{n} a_{ij} y_j\right).$$

2°. For every $\varepsilon > 0$ there exist $x \in X^\varepsilon$, a number $a^\varepsilon$, and a vector

$$y^\varepsilon = (y_1^\varepsilon, y_2^\varepsilon, \dots, y_n^\varepsilon),$$

such that for all $i = 1, 2, \dots, m$

$$x_i = a^\varepsilon \exp\left(\sum_{j=1}^{n} a_{ij} y_j^\varepsilon\right).$$

3°. For arbitrary $x \in X$ there exist a strategy $\tilde{x}$, equivalent to $x$, a number $\tilde{a}$, and a vector

$$\tilde{y} = (\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_n),$$

such that for $i \in I_x$

$$\tilde{x}_i = \tilde{a} \exp\left(\sum_{j=1}^{n} a_{ij} \tilde{y}_j\right).$$

4°. For arbitrary $x \in X$ and for every $\varepsilon > 0$ there exist a strategy $\tilde{x}$, $\varepsilon$-equivalent to $x$, a number $\tilde{a}^\varepsilon$, and a vector

$$\tilde{y}^\varepsilon = (\tilde{y}_1^\varepsilon, \tilde{y}_2^\varepsilon, \dots, \tilde{y}_n^\varepsilon),$$

such that for all $i = 1, 2, \dots, m$

$$\tilde{x}_i = \tilde{a}^\varepsilon \exp\left(\sum_{j=1}^{n} a_{ij} \tilde{y}_j^\varepsilon\right).$$

In the proof of the theorem the following lemma is used.

**Lemma.** Suppose that at the point $x^0$ the maximum is attained of the function

$$f(x) = \sum_{i=1}^{m} (x_i - x_i \ln x_i)$$

(where, for $x_i = 0$, it is assumed that $x_i \ln x_i = 0$) on the convex set $S$ of the nonnegative orthant. Denote by $I_S$ the set⋯

indices $i$ for which there exists an $x_i \in S$ such that $x_i > 0$. Then $x_i^0 > 0$ for all $i \in I_S$.

**Proof.** Suppose that $x_{i_1}^0 = 0$ for some $i_1 \in I_S$. Choose an $x^1 \in S$ such that $x_i^1 > 0$ for all $i \in I_S$. Let

$$x(\lambda) = \lambda x^1 + (1 - \lambda) x^0$$

and $\varphi(\lambda) = f(x(\lambda))$. Then

$$\varphi'(\lambda) = -\sum_{i=1}^{m} \ln x_i(\lambda)(x_i^1 - x_i^0).$$

Since $x_{i_1}^1 > x_{i_1}^0$, it follows that $\varphi'(\lambda) \to \infty$ as $\lambda \to 0$. Consequently, there exists a number $\lambda_0 > 0$ such that, for $\lambda < \lambda_0$, $f(x(\lambda)) > f(x^0)$, and hence $x^0$ is not a point of maximum of the function $f(x)$ on $S$.

**Proof of the theorem.** Suppose that at the point $x^0$ the function

$$f(x) = \sum_{i=1}^{m} (x_i - x_i \ln x_i)$$

attains its maximum on the set $X^0$. Since $X^0$ is a convex set, $x_i^0 > 0$ for all $i \in I_1$.

Denote by $v$ the value of the game $\Gamma$.

By the Kuhn–Tucker theorem, $x^0$ is the $x$-component of a saddle point of the function

$$\varphi(x, w, y) = f(x) + w \left( \sum_{i=1}^{m} x_i - 1 \right) + \sum_{j=1}^{n} y_j \left( \sum_{i=1}^{m} a_{ij} x_i - v \right)$$

in the domain $\{x_i \geq 0,\ y_j \geq 0;\ i = 1, 2, \dots, m;\ j = 1, 2, \dots, n\}$. Consequently,

$$\left. \frac{\partial \varphi(x, w, y)}{\partial x_i} \right|_{x_i = x_i^0} = 0 \quad \text{for } i \in I_1.$$

Hence

$$\ln x_i = w + \sum_{j=1}^{n} a_{ij} y_j \quad \text{for } i \in I_1,$$

and assertion 1° is proved.

Assertion 2° is proved analogously; it is sufficient only to note that $I_{X^\varepsilon} = \{1, 2, \dots, m\}$.

The proof of assertions 3° and 4° is carried out by the same scheme.

Suppose the game $\Gamma$ is the sum of matrix games $\Gamma_1, \Gamma_2, \dots, \Gamma_k$, i.e., the strategies of the first and second players are tuples $(i_1, i_2, \dots, i_k)$ and $(j_1, j_2, \dots, j_k)$ $(1 \leq i_\nu \leq m_\nu;\ 1 \leq j_\nu \leq n_\nu)$, where $i_\nu$ and $j_\nu$ are certain strategies of the first and second players in the game $\Gamma_\nu$, and

$$a_{ij} = \sum_{\nu=1}^{k} b_{i_\nu j_\nu}^{\nu},$$

where

$$B_\nu = \|b^\nu_{i_\nu j_\nu}\| \quad (i_\nu = 1, 2, \dots, m_\nu;\ j_\nu = 1, 2, \dots, n_\nu)$$

is the matrix of the game $\Gamma_\nu$. Then it is obvious that, for any mixed strategies $x$ and $y$ in the game $\Gamma$, there exist equivalent strategies $\tilde{x}$ and $\tilde{y}$, such that

$$\tilde{x}_i = \prod_{\nu=1}^{k} x^\nu_{i_\nu}, \qquad \tilde{y}_j = \prod_{\nu=1}^{k} y^\nu_{j_\nu},$$

where

$$i = (i_1, i_2, \dots, i_k), \qquad j = (j_1, j_2, \dots, j_k),$$

and

$$x^\nu = (x^\nu_1, \dots, x^\nu_{m_\nu}), \qquad y^\nu = (y^\nu_1, \dots, y^\nu_{n_\nu})$$

are certain mixed strategies of the first and second players in the game $\Gamma_\nu$. In other words, strategies equivalent to arbitrary mixed strategies of the first and second players in the game $\Gamma$ can be obtained by choosing independently strategies in the games $\Gamma_\nu$.

As follows from Theorem 1, an analogous result holds for the strategies of the first player in games that are the sum of several matrix games with constraints, i.e., for games differing from an ordinary sum of games in that not all tuples $(i_1, i_2, \dots, i_k)$ of strategies in the games $\Gamma_\nu$ are strategies of the first player in the game $\Gamma$. Namely, the following holds.

**Theorem 2.** Let $I$—the set of pure strategies of the first player in the game $\Gamma$ —be contained in the set of tuples $(i_1, i_2, \dots, i_k)$ $(1 \le i_\nu \le m_\nu)$, where $i_\nu$ is a certain pure strategy in the game $\Gamma_\nu$, and

$$a_{ij} = \sum_{\nu=1}^{k} b^\nu_{i_\nu j},$$

where

$$B_\nu = \|b^\nu_{i_\nu j}\| \quad (i_\nu = 1, 2, \dots, m_\nu;\ j = 1, 2, \dots, n)$$

is the matrix of the game $\Gamma_\nu$.

Then:

1°. There exist $x \in X^0$, a number $a$, and mixed strategies of the first player $x^\nu$ in the games $\Gamma_\nu$ ($\nu = 1, 2, \dots, k$), such that for $i \in I_1$ one has

$$x_i = a \prod_{\nu=1}^{k} x^\nu_{i_\nu}.$$

2°. For any $\varepsilon > 0$ there exist $x \in X^\varepsilon$, a number $a^\varepsilon$, and mixed strategies of the first player $x^\nu$ in the games $\Gamma_\nu$ ($\nu = 1, 2, \dots, k$), such that for all $i \in I$ one has

$$x_i = a^\varepsilon \prod_{\nu=1}^{k} x^\nu_{i_\nu}.$$

3°. For an arbitrary $x \in X$ there exist a strategy $\tilde{x}$, equivalent to $x$, a number $\tilde{a}$, and mixed strategies of the first player $x^\nu$ in the games $\Gamma_\nu$, such that for $i \in I_x$ one has

$$\tilde{x}_i = \tilde{a} \cdot \prod_{\nu=1}^{k} x^\nu_{i_\nu}.$$

4°. For an arbitrary $x \in X$ and for any $\varepsilon > 0$ there exist a strategy $\tilde{x}$, $\varepsilon$-equivalent to $x$, a number $\tilde{a}^\varepsilon$, and mixed strategies of the first player $x^\nu$ in the games $\Gamma_\nu$, such that for all $i \in I$ one has

$$\tilde{x}_i = \tilde{a}^\varepsilon \prod_{\nu=1}^{k} x^\nu_{i_\nu}.$$

To prove this theorem it is enough to put

$$x^\nu_{i_\nu} = \exp\left(\sum_{j=1}^{n} b^\nu_{i_\nu j} y_j\right) \Big/ \sum_{i_\nu=1}^{m_\nu} \exp\left(\sum_{j=1}^{n} b^\nu_{i_\nu j} y_j\right),$$

where $y = (y_1, y_2, \dots, y_n)$ is the vector whose existence is proved in Theorem 1.

Let $\Gamma$ be the Blotto game (1), i.e., the strategies of the first and second players in it are respectively sets $i = (i_1, i_2, \dots, i_k)$ and $j = (j_1, j_2, \dots, j_k)$ of nonnegative integers satisfying the conditions

$$\sum_{\nu=1}^{k} i_\nu = A, \qquad \sum_{\nu=1}^{k} j_\nu = B,$$

and the elements of the matrix of this game are representable in the form:

$$a_{ij} = \sum_{\nu=1}^{k} b^{\nu}_{i_\nu j_\nu}.$$

It is clear that this game is the sum of the matrix games $\Gamma_\nu$, whose matrices are $B_\nu = \|b^{\nu}_{i_\nu j_\nu}\|$ ($i_\nu = 0, 1, \dots, A;\ j_\nu = 0, 1, \dots, B$), with restrictions. Therefore Theorem 2 holds for the game $\Gamma$. Thus, in Blotto games, a mixed strategy equivalent to an arbitrary mixed strategy of the first player $x$ can be obtained by choosing the strategies $i_\nu$ in the games $\Gamma_\nu$ independently and discarding those sets $(i_1, i_2, \dots, i_k)$ which are not contained in the set $I_x$.

Consider now the game $\Gamma$ in which the pure strategies of the first player are some of the sets $i = (i_1, i_2, \dots, i_k)$ $(1 \le i_\nu \le m_\nu)$, and the elements of the game matrix have the form

$$a_{ij} = b^{1}_{i_1 j_1} + \sum_{\nu=2}^{k} b^{\nu}_{i_{\nu-1} i_\nu j_\nu}.$$

As follows from Theorem 1, in this case for any mixed strategy $x$ of the first

there exists for the player in the game $\Gamma$ a strategy $\tilde{x}$, equivalent to $x$, such that for $i \in I_x$ one has

$$\tilde{x}_i = a x^{1}_{i_1} \prod_{\nu=2}^{k} x^{\nu}_{i_{\nu-1} i_\nu},$$

where

$$\sum_{i_1=1}^{m_1} x^{1}_{i_1} = 1 \quad \text{and} \quad \sum_{i_\nu=1}^{m_\nu} x^{\nu}_{i_{\nu-1} i_\nu} = 1 \quad (\nu = 2, 3, \dots, k).$$

Thus, a strategy equivalent to an arbitrary mixed strategy $x$ of the first player in the game $\Gamma$ can be obtained by modeling a certain Markov chain and discarding those sets $(i_1, i_2, \dots, i_k)$ which are not contained in the set $I_x$.

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## CITED LITERATURE

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*Note: Figure translations are in progress. See original paper for figures.*