CUBATURE FORMULAS WITH PRESCRIBED DERIVATIVES IN THE PERIODIC CASE

MATHEMATICS

1969

SovietRxiv

Abstract

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UDC 518:517.392

MATHEMATICS

T. I. KHAITOV

CUBATURE FORMULAS WITH PRESCRIBED DERIVATIVES IN THE PERIODIC CASE

(Presented by Academician S. L. Sobolev, 19 V 1969)

In the present paper, the results of S. L. Sobolev on the integration of periodic functions from ([1–6]) are generalized to the case of cubature formulas that are n-dimensional analogues of the Hermite quadrature formula.

Recall that a function $f(x)$ is called periodic with period matrix H, $|H| = 1$, if for any integer vector B and all x one has $f(x) = f(x + H\beta)$.

Identifying in Euclidean space E_n points that differ from one another by a period, we obtain the n-dimensional torus Ω_0. This torus is called the fundamental domain. The condition of fundamentality can be written in the form

$$\sum_{\beta} \mathcal{E}_{\Omega_0}(x - H\beta) = 1.$$

Here $\mathcal{E}_{\Omega_0}(x)$ is the characteristic function of the domain Ω_0.

To each function $\varphi(x) \in K$ from the basic space ([1,8]) one can associate the function $\mathring{\varphi}(x) \in \mathring{K}$, defined on the torus, by the formula

$$\mathring{\varphi}(x) = \varphi(x) * \Phi^{(0)}(x) = \sum_{\beta} \varphi(x + \beta),$$

where

$$\Phi^{(0)}(x) = \sum_{\beta} \delta(x + \beta).$$

If $l(x)$ is a periodic functional in E_n, then one can associate with it a functional $\hat{l}(x)$, defined on the torus by the formula

$$\int_{E_n} l(x)\varphi(x)\, dx = \int_{\Omega_0} \hat{l}(x)\mathring{\varphi}(x)\, dx.$$

These most important concepts are set forth fully, for example, in ([1]).

In our case the error functional has the form

$$l^{(t)}(x) = 1 - \sum_{|\alpha|\le 2t} (-1)^{|\alpha|} C^{(\alpha)} \Phi_H^{(\alpha)}(x);$$

$$(l^{(t)}, \varphi) = \int_{\Omega_0} \varphi(x)\,dx - \sum_{|\alpha|\le 2t} C^\alpha D^\alpha \varphi(0); \tag{1}$$

$$2m > n; \qquad 2t \le m - [n/2] - 1; \qquad (l^{(t)}, 1) = 0. \tag{2}$$

Here

$$\Phi_H^{(\alpha)}(x) = \sum_\beta D^\alpha \delta(x - H\beta); \qquad \Phi_H^{(\dot{0})}(x) = \sum_\beta \delta(x - H\beta); \qquad |\alpha| = \alpha_1 +$$

$+\alpha_2 + ... + \alpha_n$; $\quad D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}$; $\quad C^{(\alpha)}$ are the coefficients of the cu-

cubature formula. From $(l^{(t)}(x),\, 1) = 0$ it follows that $C^{(0)} = 1$. The following norms in $\widetilde{L}_2^m(H)$ are used:

$$\widetilde{L}_2^m(H) = \left\{ f:\ f(x) = \sum_\beta f_\beta e^{2\pi i(\beta H^{-1}, x)},\ \|f\|^2_{\widetilde{L}_2^m} = (2\pi)^{2m} \sum_\beta |f_\beta|^2 |\beta H^{-1}|^{2m} \right\}, \tag{3}$$

and also

$$\widetilde{L}_2^m(H) = \left\{ f:\ f(x) = f(x + H\beta),\ \|f\|^2_{\widetilde{L}_2^m} = \int_{\Omega_0} \sum_{|\alpha|=m} \frac{|\alpha|!}{\alpha!} (D^\alpha f)^2\, dx \right\},$$

$\alpha! = \alpha_1! \alpha_2! \cdots \alpha_n!$; f_β are the Fourier coefficients of the function $f(x)$. Weighted cubature formulas in $\widetilde{L}_2^m$ with norm (3) were considered in (7).

Theorem. If $l^{(t)}(x)$ is a functional of the form (1), (2) from $\widetilde{L}_2^{m*}$, then

$$\|l^{(t)}\|^2_{\widetilde{L}_2^{m*}} = (2\pi)^{-2m} \sum_{\beta \ne 0} |\beta H^{-1}|^{-2m} \Psi^2_{H^{-1}}(C_{H^{-1}}, \beta), \tag{4}$$

$$\Psi_{H^{-1}}(C_{H^{-1}},\beta)=\sum_{s=0}^{t}\sum_{|\alpha|=2s}(-1)^s(2\pi)^{2s}(\beta H^{-1})^\alpha C^{(\alpha)}_{H^{-1}}. \tag{5}$$

Proof. We estimate $(l^{(t)},\varphi)$ for $\varphi\in\widetilde{L}_2^m$, applying the Cauchy inequality:

$$\begin{aligned}|(l^{(t)},\varphi)| &= \left|(l^{(t)},\sum_\beta \varphi_\beta e^{2\pi i(\beta H^{-1},x)})\right| \\ &= \left|\sum_{\beta\neq 0}\varphi_\beta\left(-\sum_{|\alpha|\le 2t}(2\pi i)^{|\alpha|}(\beta H^{-1})^\alpha C^{(\alpha)}_{H^{-1}}\right)\right| \\ &= \left|\sum_{\beta\neq 0}(2\pi)^m|\beta H^{-1}|^m\varphi_\beta\left(-(2\pi)^{-m}\sum_{|\alpha|\le 2t}(2\pi i)^{|\alpha|}(\beta H^{-1})^\alpha C^{(\alpha)}_{H^{-1}}|\beta H^{-1}|^{-m}\right)\right| \\ &\le \left[(2\pi)^{-2m}\sum_{\beta\neq 0}|\beta H^{-1}|^{-2m}\left|\sum_{|\alpha|\le 2t}(2\pi i)^{|\alpha|}(\beta H^{-1})^\alpha C^{(\alpha)}_{H^{-1}}\right|^2\right]^{1/2}\|\varphi\|_{\widetilde{L}_2^m}.\end{aligned} \tag{6}$$

Put

$$u(x)=-(2\pi)^{-2m}\sum_{\beta\neq 0}\left\{|\beta H^{-1}|^{-2m}\sum_{|\alpha|\le 2t}(2\pi i)^{|\alpha|}(\beta H^{-1})^\alpha C^{(\alpha)}_{H^{-1}}\times e^{2\pi i(\beta H^{-1},x)}\right\};$$

then for $u(x)$ equality is attained in (6):

$$|(l^{(t)},u)|=\left[(2\pi)^{-2m}\sum_{\beta\neq 0}|\beta H^{-1}|^{-2m}\left|\sum_{|\alpha|\le 2t}(2\pi i)^{|\alpha|}(\beta H^{-1})^\alpha C^{(\alpha)}_{H^{-1}}\right|^2\right]^{1/2}\|u\|_{\widetilde{L}_2^m}. \tag{7}$$

If in (7) β is replaced by $-\beta$, then the terms with odd $|\alpha|$ cancel each other. The theorem is proved.

Put $y=Hx$ or $x=H^{-1}y$; then

$$\|l^{(t)}\|^2_{\widetilde{L}_2^{m*}}=(2\pi)^{-2m}\sum_{\beta\neq 0}(\beta,\beta^*)^{-m}\Psi^2(c,\beta), \tag{8}$$

$$\Psi(c,\beta)=1-\sum_{s=1}^{t}\sum_{|\alpha|=2s}(-1)^{s-1}(2\pi)^{2s}\beta^{\alpha}C^{(\alpha)}. \tag{9}$$

Here β^* is conjugate to β, and $\beta^\alpha = \beta_1^{\alpha_1}\beta_2^{\alpha_2}\cdots\beta_n^{\alpha_n}$. Put $t \geq n$, $r_0 = [t/n]$, $p = t - nr_0$, $B = \{\beta : \ |\beta_i| \leq r_0 \ (i = 1, 2, \dots, n)\}$, and solve the following problem.

Problem. Find the coefficients $C^{(\alpha)}$ for (9) so that $\Psi(c,\beta) = 0$ for all integer $\beta \in B$.

The solution of this problem can be represented in the form

$$\Psi(c,\beta)=r_0!^{-2n}(r_0+1)^{-2p}\prod_{s=1}^{n}\prod_{k=1}^{r_0}(\beta_s^2-k^2)\prod_{s=1}^{p}[\beta^2-(r_0+1)^2],$$

which grows as $\beta \to \infty$. Applying the inequality ((9), p. 29), we obtain—

we have the following estimate:

$$|\Psi(c,\beta)|\leqslant r_0!^{-2n}(r_0+1)^{-2p}n^{-nr_0}p^{-p}\left(\sum_{s=1}^{n}\beta_s^2\right)^{t}, \qquad \beta\notin B. \tag{10}$$

Substituting (10) into (8), we obtain

$$\|l^{(t)}\|^2_{\widetilde{L}_2^{m*}}\leqslant(2\pi)^{-2m}r_0!^{-4m}(r_0+1)^{-4p}n^{-2nr_0}p^{-2p}\sum_{\beta\notin B}(\beta,\beta^*)^{-m+2t}. \tag{11}$$

Let now a periodic lattice be given with period matrix H^{-1}, $|H^{-1}| = 1$. On the basis of (2) (pp. 64-78), we may regard H^{-1} as a triangular matrix.

Put

$$B_{H^{-1}}=\left\{\beta H^{-1}:\left|\sum_{i=j}^{n}h_{ij}\beta_i\right|\leqslant r_1\ (j=1,2,\dots,n)\right\}.$$

Renumber all $\beta H^{-1} \in B_{H^{-1}}$ and, for convenience, introduce the notation

$$\beta^{(k)}H^{-1}=\gamma^{(k)}\in B_{H^{-1}};\qquad a_s=\prod_{k=1}^{\sigma_s}\gamma_s^{(k)};\qquad \gamma_s^{(k)}=\sum_{i=s}^{n}h_{is}\beta_i^{(k)};\qquad |\gamma_s^{(k)}|\leqslant r_1;$$

$s = 1, 2, \dots, n$; $k = 1, 2, \dots, \sigma_s$; h_{ij} is an element of the matrix H^{-1}, r_1 is chosen so that the condition $\sum_{k=1}^{n} \sigma_k = t$ is satisfied.

Solving the preceding problem for this case, we shall have the following estimates for (5) and (4):

$$\Psi_{H^{-1}}(C_{H^{-1}}, \beta) = \prod_{s=1}^{n} \prod_{k=1}^{\sigma_s} \frac{\gamma_s^2 - \gamma_s^{(k)2}}{a_1^2 a_2^2 \dots a_n^2}; \qquad |\Psi_{H^{-1}}| \leqslant \frac{\sigma_0^{2t} t^{-2t}}{a_1^2 a_2^2 \dots a_n^2} |\beta H^{-1}|^{2t};$$

$$\|l^{(t)}\|^2_{\widetilde{L}_2^{m*}} \leqslant \frac{(2\pi)^{-2m} \sigma_0^{4t} t^{-4t}}{a_1^4 a_2^4 \dots a_n^4} \sum_{\beta H^{-1} \notin B_{H^{-1}}} |\beta H^{-1}|^{-2m+4t}; \qquad \sigma_0 = \max(\sigma_1, \sigma_2, \dots, \sigma_n). \tag{12}$$

We inscribe a ball in B and $B_{H^{-1}}$. The radius of the ball is, obviously, respectively equal to r_0 and r_1. The estimate

$$\sum_{\beta \notin B} r^{-2m+4t} \leqslant \frac{\varkappa_n (1 + o(1))}{(r_0 + 1)^{2m-4t-n}}, \qquad \varkappa_n = \frac{2\pi^{n/2}}{(2m - 4t - n)\Gamma(n/2)}, \qquad r^2 = \sum_{s=1}^{n} \beta_s^2,$$

belongs to L. V. Voĭtishek. Substituting this into (11) and (12), we obtain an estimate of the norm of the functional $l^{(t)}(x)$, respectively for the case of rectangular lattices and lattices with matrix H^{-1}:

$$\|l^{(t)}\|^2_{\widetilde{L}_2^{m*}} \leqslant \frac{\varkappa_n n^{-2nr_0} p^{-2p} r_0!^{-4n}}{(2\pi)^{2m} (r_0 + 1)^{2m-4t+4p-n}} + O\left(\frac{1}{(r_0 + 1)^{2m-4t-n}}\right), \tag{13}$$

$$\|l^{(t)}\|^2_{\widetilde{L}_2^{m*}} \leqslant \frac{\varkappa_n \sigma_0^{4t} t^{-4t}}{a_1^4 a_2^4 \dots a_n^4 (2\pi)^{2m} (r_1 + 1)^{2m-4t-n}} + O\left(\frac{1}{(r_1 + 1)^{2m-4t-n}}\right).$$

Estimate (13) can be transformed into the form

$$\|l^{(t)}(x)\|^2_{\widetilde{L}_2^{m*}} \leqslant \frac{r_0^{-4n} (r_0 + 1)^{-2m-4p+4t+n}}{(2m - 4t - n) n^{2nr_0}} \|l^{(0)}(x)\|^2_{\widetilde{L}_2^{m*}}, \tag{14}$$

where $l^{(0)}(x)$ is obtained from $l^{(t)}(x)$ for $t = 0$.

Example. Put in (14) $n = 2,\ t = 2,\ m = 6,\ r_0 = 1,\ p = 0$; then

$$\|l^{(2)}(x)\|^2_{\widetilde{L}_2^{6*}} \ll \frac{1}{128} \|l^{(0)}(x)\|^2_{\widetilde{L}_2^{m*}}.$$

Now let

$$I = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} e^{\sin(x+y)}\,dx\,dy.$$

Computations with step $h = \pi/3$ gave the values

$$\|l^{(0)}\|_{\widetilde{L}_2^{6*}} = 0.0000450\ldots, \qquad \|l^{(2)}\|_{\widetilde{L}_2^{6*}} = 0.000000041\ldots.$$

Remark. In cases where computing the derivatives of $\varphi(x)$ does not present difficulties, it is advantageous to apply the cubature formula considered in this note.

The author expresses his deep gratitude to the supervisor of this work, S. L. Sobolev.

Institute of Mathematics
Siberian Branch of the Academy of Sciences of the USSR
Novosibirsk

Received
13 V 1969

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