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Abstract

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ON THE LIMITING BEHAVIOR OF AN INVARIANT MEASURE UNDER SMALL PERTURBATIONS OF A DYNAMICAL SYSTEM

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Let a vector field $\{b^i(x)\} = b(x)$ of class $C^1(S)$ be given on a connected compact manifold S of dimension n. Consider on S the dynamical system

$$\dot{x} = b(x) \tag{1}$$

and the Markov process $X^\varepsilon = \{x_t^\varepsilon, P_x^\varepsilon\}$ (see ([1,2])), governed by the differential operator

$$L^\varepsilon u(x) = \frac{\varepsilon^2}{2\sqrt{a(x)}} \sum_{i=1}^n \frac{\partial}{\partial x^i} \sum_{j=1}^n a^{ij}(x) \frac{\partial}{\partial x^j}(a(x)u(x)) + \sum_{i=1}^n b^i(x) \frac{\partial u}{\partial x^i},$$

$a^{ij}(x) \in C^1(S)$, $a(x) = \det\{a^{ij}(x)\}^{-1}$, and the quadratic form $\sum_{i,j=1}^n a^{ij}(x)\lambda_i\lambda_j$ is positive definite. Since the manifold S is compact and the operator L^ε is nondegenerate for $\varepsilon > 0$, the process X^ε has a unique stationary distribution (normalized invariant measure) μ^ε. The dynamical system (1), obtained from X^ε for $\varepsilon = 0$, also has a normalized invariant measure, but, generally speaking, not a unique one. It is natural to expect that μ^ε converges as $\varepsilon \to 0$ to some invariant measure of system (1), but the question arises: to which one exactly? In the present note we study the question of on what set the distribution μ^ε will concentrate as $\varepsilon \to 0$. In the one-dimensional case, as well as in the case when $b(x)$ is a potential field and $\{a^{ij}(x)\}$ is the identity matrix, an explicit expression for its density may be used to study the limiting behavior of the invariant measure of the process X^ε (see ([1,3–5])). In the potential case the measure μ^ε as $\varepsilon \to 0$ concentrates at the point of the absolute minimum of the potential. Theorem 1 below shows what must replace the potential in the general case.

Denote by H the set of continuous, piecewise continuously differentiable functions φ_s, defined on intervals of the real axis of the form $[0, T]$, with values in S. On the elements of H consider the functional

$$I(\varphi) = \int_0^T \sum_{i,j=1}^{n} a_{ij}(\varphi_s)(\dot{\varphi}_s^i - b^i(\varphi_s))(\dot{\varphi}_s^j - b^j(\varphi_s))\, ds,$$

where $\{a_{ij}(x)\}$ is the matrix inverse to $\{a^{ij}(x)\}$, and put

$$V(x, y) = \inf_{\varphi(0)=x,\ \varphi(T)=y} I(\varphi).$$

The function $V(x, y)$ is continuous in both arguments.

Introduce an equivalence relation for points of the manifold S: $x \sim y$ if $V(x, y) = V(y, x) = 0$. This relation in fact does not depend on the matrix $\{a^{ij}(x)\}$, but is determined only by the structure of the dynamical system (1).

Suppose that the following condition is fulfilled:

Condition A. On the manifold S there exists a finite number of compact sets $K_1, K_2, \dots, K_l$ such that: 1) for any points x, y of one compact set ⋯

the relation $x \sim y$ holds; 2) if $x \in K_i$, and $y \notin K_i$, then $x \nsim y$; 3) every α- or ω-limit set (see ([6])) of system (1) belongs entirely to one of the K_i.

Let us note that, when condition 2) is satisfied, a limit set cannot intersect the compact set K_i without belonging to it entirely.

It follows from condition A that a point which does not belong to any of the K_i cannot be equivalent to any other point.

We shall call K_i a **stable set** if $V(x, y) > 0$ for $x \in K_i,\ y \notin K_i$. The property of stability also depends only on the structure of system (1).

It is easy to see that the function $V(x, y)$ takes one and the same value for all $x \in K_i,\ y \in K_j$; denote this value by V_{ij}.

We shall call an ***i*-graph** a graph consisting of $l - 1$ arrows $(j \to k)$, where $j, k = 1, 2, \dots, l,\ k \neq j$, satisfying the following conditions: 1) j runs through all values from 1 to l, except i; 2) the graph has no closed cycles.

Denote the set of all i-graphs by G_i; for its elements we shall use the letter g.

Put

$$V_i = \min_{g \in G_i} \sum_{(j \to k) \in g} V_{jk}.$$

It is easy to prove that, if $V_i = \min(V_1, V_2, \dots, V_l)$, then the compact set K_i is stable.

Theorem 1. *Suppose condition A is satisfied. Denote by M the set of those i, $1 \leq i \leq l$, for which $V_i = \min(V_1, \dots, V_l)$. If B is an arbitrary open set containing $\bigcup_{i \in M} K_i$, then*

$$\lim_{\varepsilon \to 0} \mu^\varepsilon(B) = 1.$$

Theorem 2. *Suppose condition A is satisfied, the set M consists of a single element i_0, and there exists only one normalized invariant measure μ_0 of system (1), concentrated on K_{i_0}. Then the measure μ^ε converges weakly to μ_0 as $\varepsilon \to 0$.*

For the proof of Theorem 1, surround each compact set K_i by a pair of neighborhoods d_i, D_i with smooth boundaries γ_i, Γ_i, respectively, and let $\bar{d}_i \cup \gamma_i \subset D_i$. Denote $\gamma = \bigcup_i \gamma_i$, $\Gamma = \bigcup_i \Gamma_i$. Introduce the random times $\tau_0 = 0$,

$$\sigma_n = \inf\{t \geq \tau_{n-1} : x_t^\varepsilon \in \Gamma\}, \qquad \tau_n = \inf\{t \geq \sigma_n : x_t^\varepsilon \in \gamma\}.$$

Consider the Markov chain $z_n = x^\varepsilon_{\tau_n}$ on the phase space γ. It is known ([7]) that the invariant measure μ^ε of the process X^ε can be expressed, up to a factor, in terms of the invariant measure ν^ε of the chain $\{z_n\}$ by the formula

$$\mu^\varepsilon(B) = \int_\gamma \nu^\varepsilon(dy)\, M_y^\varepsilon \int_0^{\tau_1} \chi_B(x_t^\varepsilon)\, dt \tag{2}$$

(M_y^ε is the mathematical expectation corresponding to the trajectory x_t^ε issuing from the point y at $t = 0$).

The estimates for the invariant measure ν^ε of the chain $\{z_n\}$ and for the mathematical expectation standing under the integral in (2) which are given below are proved with the aid of Lemmas 1, 2 of ([8]) and the following generalization of Lemma 3 of the same work:

Lemma 1. *For any choice of the neighborhoods d_i, D_i there exist constants $T_0, C > 0$ such that, for all sufficiently small ε and for all $x \in \Gamma$, the inequality*

$$P_x^\varepsilon\{\tau_1 > T\} \leq \exp[-C(T - T_0)/2\varepsilon^2].$$

The following lemma gives an estimate of the transition probabilities.

Lemma 2. For any $h > 0$ one can indicate an $r > 0$ such that, for any choice of neighborhoods d_i, D_i inside the r-neighborhoods K_i, $i = 1, 2, \dots, l$, there exists $\varepsilon_0 > 0$ such that the probability of transition of the chain $\{z_n\}$ in $s = l - 1$ steps, for $\varepsilon < \varepsilon_0$, $x \in \gamma_i$, satisfies the inequalities

$$\exp\left[\frac{-V_{ij} - h}{2\varepsilon^2}\right] < P^{(s)}(x, \gamma_j) < \exp\left[\frac{-V_{ij} + h}{2\varepsilon^2}\right].$$

For estimating the invariant measure ν^ε, the following lemmas are used.

Lemma 3. Suppose there is a Markov chain with l states for which the transition probabilities $p_{ij} > 0$ for $j \neq i$. Then the stationary distribution of this chain, up to a factor, is $(Q_1, \dots, Q_l)$, where

$$Q_i = \sum_{g \in G_i} \prod_{(j \to k) \in g} p_{jk}.$$

The same is true if, instead of p_{jk}, one takes the transition probabilities in s steps, $s > 1$.

Lemma 4. Denote by $\underline{Q}_i, \overline{Q}_i$ the quantities obtained by substituting in formula (3), in place of p_{ij}, the lower and upper estimates for the transition probabilities in s steps $p^{(s)}(x, \gamma_i)$, $x \in \gamma_i$ (see Lemma 2). Then the ratio of the values of the invariant measure ν^ε for the sets Γ_i and Γ_j lies between $\underline{Q}_i : \overline{Q}_j$ and $\overline{Q}_i : \underline{Q}_j$.

The mathematical expectations under the integral sign in equality (2) are estimated with the help of the following lemmas.

Lemma 5. If K_i is a stable set, then for any of its neighborhoods D_i one can choose a smaller neighborhood d_i with smooth boundary, $\gamma_i, d_i \cup \gamma_i \subset D_i$, and a constant $a > 0$, such that for all sufficiently small ε

$$M_x^\varepsilon \sigma_1^3 > \exp(a/2\varepsilon^2), \qquad x \in \gamma_i.$$

Lemma 6. For any $a > 0$ there exists an $r > 0$ such that, if D_i is contained in the r-neighborhood of K_i, then for all sufficiently small ε

$$M_x^\varepsilon \sigma_1 < \exp(a/2\varepsilon^2), \qquad x \in \gamma_i.$$

Moreover, it follows from Lemma 1 that, for ε less than some $\varepsilon_0 > 0$,

$$M_x^\varepsilon \tau_1 \leq T_0 + 2\varepsilon^2/c < T_1 = T_0 + 2\varepsilon_0^2/c, \qquad x \in \Gamma.$$

The proof of Theorem 1 is obtained from the lemmas as follows. First choose a positive

$$h < \frac{1}{2l}\left[\min_{i \notin M} V_i - \min_{1 \leq i \leq l} V_i\right].$$

Next choose positive r in accordance with Lemma 2 and fix neighborhoods D_i of those compact sets K_i for which V_i assumes its smallest value ($i \in M$) inside the r-neighborhoods of these compact sets, in such a way that they lie entirely inside the set B. Then, in accordance with Lemma 5, choose their smaller neighborhoods d_i and a constant $a > 0$; using Lemma 6 with this constant, we find how small the remaining neighborhoods D_i and d_i must be, and fix these neighborhoods. By virtue of the estimates of Lemma 2, the ratios $Q_i : Q_j$ for $i \notin M$, $j \in M$ will tend to zero as $\varepsilon \to 0$ no more slowly than $\exp(-h/\varepsilon^2)$; therefore, by Lemma 4,

$$\nu^\varepsilon\left(\bigcup_{i\notin M}\gamma_i\right) = O\left(\exp(-h/\varepsilon^2)\right), \qquad \nu^\varepsilon\left(\bigcup_{i\in M}\gamma_i\right) \to 1 \quad \text{as } \varepsilon \to 0.$$

Hence,

$$\int_\gamma \nu^\varepsilon(dy)\, M_y^\varepsilon \int_0^{\tau_1} \chi_B(x_t^\varepsilon)\,dt \geq \sum_{i\in M}\int_{\gamma_i}\nu^\varepsilon(dy)\, M_y^\varepsilon \int_0^{\sigma_1}\chi_B(x_t^\varepsilon)\,dt \geq$$

$$\geq \sum_{i\in M}\int_{\gamma_i}\nu^\varepsilon(dy)\, M_y^\varepsilon \sigma_1 \geq \text{const}\cdot\exp(a/2\varepsilon^2).$$

for sufficiently small ε. At the same time

$$\int_\gamma \nu^\varepsilon(dy)\, M_y^\varepsilon \int_0^{\tau_1}\chi_{S\setminus B}(x_t^\varepsilon)\,dt = \sum_{i\in M}\int_{\gamma_i}\nu^\varepsilon(dy)\, M_y^\varepsilon\int_0^{\sigma_1}\chi_{S\setminus B}(x_t^\varepsilon)\,dt +$$

$$+\sum_{i\notin M}\int_{\gamma_i}\nu^\varepsilon(dy)\, M_y^\varepsilon\int_0^{\sigma_1}\chi_{S\setminus B}(x_t^\varepsilon)\,dt + \int_\gamma \nu^\varepsilon(dy)\, M_y^\varepsilon\int_{\sigma_1}^{\tau_1}\chi_{S\setminus B}(x_t^\varepsilon)\,dt.$$

The first integral is equal to zero; the second, for sufficiently small ε, does not exceed

$$\sum_{i\notin M}\int_{\gamma_i}\nu^\varepsilon(dy)\, M_y^\varepsilon\sigma_1 \leq \text{const}\cdot\exp\left\{\frac{a-2h}{2\varepsilon^2}\right\};$$

in the third,

$$M_y^\varepsilon\int_{\sigma_1}^{\tau_1}\chi_{S\setminus B}(x_t^\varepsilon)\,dt \leq M_y^\varepsilon[\tau_1-\sigma_1] = M_y^\varepsilon M_{x_{\sigma_1}^\varepsilon}^\varepsilon \tau_1 < T_1.$$

This means that $\mu^\varepsilon(S\setminus B) = O(\mu^\varepsilon(B))$ as $\varepsilon \to 0$, i.e. $\mu^\varepsilon(B) \to 1$.

To prove Theorem 2, one verifies that if μ_0 is a weak limit of a subsequence of the measures μ^ε, then the measure μ_0 is invariant.

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Note: Figure translations are in progress. See original paper for figures.