ON THE EQUIVALENCE OF MEASURES CORRESPONDING TO GAUSSIAN VECTOR-VALUED FUNCTIONS

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Abstract

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ON THE EQUIVALENCE OF MEASURES CORRESPONDING TO GAUSSIAN VECTOR-VALUED FUNCTIONS

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In what follows $\mathcal{H}$ is a Hilbert space; $(\cdot,\cdot)$ and $\|\cdot\|$ are the scalar product and norm in $\mathcal{H}$; $\mathcal{G}(\mathcal{H})$ is the Hilbert space of Hilbert–Schmidt operators with scalar product

$$(A, B)_{G(\mathcal{H})} = \mathrm{Sp}(AB^*);$$

$K(\mathcal{H})$ is the set of nuclear operators; $\mathcal{G}_+(\mathcal{H})$ and $K_+(\mathcal{H})$ are the corresponding subsets of positive definite operators.

Let $(\Omega, \mathfrak{B})$ be a measurable space; $\mathcal{P}$ a probability measure on $\mathfrak{B}$. An $\mathcal{H}$-valued random Gaussian function $\xi(t)$, $t \in T$, determines the mathematical expectation $m(t)$ and the correlation function $R(s,t)$:

$$M(x, \xi(t)) = (x, m(t)), \qquad M(x, \xi(s) - m(s))(y, \xi(t) - m(t)) = (x, R(s,t)y),$$

$x, y \in \mathcal{H}$; $m(t)$ is an $\mathcal{H}$-valued function; $R(s,t)$ is a $K(\mathcal{H})$-valued positive definite function; $\widetilde{R}(s,t) = R(s,t) + m(s) \otimes m(t)$.

1. Let $R(s,t)$, $s,t \in T$, be a $K(\mathcal{H})$-valued positive definite function; H a Hilbert space of $\mathcal{H}$-valued functions defined on T; $\langle\cdot,\cdot\rangle$ the scalar product in H.

Definition. The function $R(s,t)$, $s,t \in T$, is called the **reproducing kernel of the space** H if, for any $t \in T$, $x \in \mathcal{H}$, a) the function $R(\cdot,t)x \in H$; b) $\langle R(\cdot,t)x, m(\cdot)\rangle = (x, m(t))$.

The space H is uniquely determined by the function $R(s,t)$ as the linear hull of the functions $m_{t,x}(\cdot) = R(\cdot,t)x$, closed with respect to the scalar product

$$\langle R(\cdot,s)x, R(\cdot,t)y\rangle = (x, R(s,t)y)$$

(cf. (1)).

Let $R_1(s,t)$ and $R_2(s,t)$, $s,t \in T$, be two $K(\mathcal{H})$-valued kernels; in the tensor product $\mathcal{H} \otimes \mathcal{H}$ consider the operators

$$\Phi(s,u;t,v)(x \otimes y) = R_1(s,t)x \otimes R_2(u,v)y,$$

$s,u,t,v \in T, \quad x,y \in \mathcal{H}$. $\Phi(s,u;t,v)$, $(s,u),(t,v) \in T \times T$, is a $K(\mathcal{H} \otimes \mathcal{H})$-valued positive definite function. We denote the kernel $\Phi(s,u;t,v)$ by $R_1(s,t) \otimes R_2(u,v)$. In accordance with what was said above, the space $H(R_1 \otimes R_2)$ is spanned by the $\mathcal{H} \otimes \mathcal{H}$-valued functions of the form

$$R_1(\cdot,t)x \otimes R_2(\cdot,v)y;$$

if to each such function one assigns the element $R_1(\cdot,t)x \otimes R_2(\cdot,v)y$ of $H(R_1) \otimes H(R_2)$, then we obtain an isomorphism

$$H(R_1 \otimes R_2) \simeq H(R_1) \otimes H(R_2);$$

this circumstance facilitates finding $H(R_1 \otimes R_2)$ when $H(R_1)$ and $H(R_2)$ are known. To every element of the form $x \otimes y \in \mathcal{H} \otimes \mathcal{H}$ there corresponds an operator $A_{x\otimes y} \in \mathcal{G}(\mathcal{H})$, defined by the formula

$$A_{x\otimes y} z = (y,z)x, \quad z \in \mathcal{H}.$$

This correspondence extends to an isomorphism between the spaces $\mathcal{H} \otimes \mathcal{H}$ and $\mathcal{G}(\mathcal{H})$. Thus we may regard functions from $H(R_1 \otimes R_2)$ as $\mathcal{G}(\mathcal{H})$-valued operator functions on $T \times T$.

Let $\xi(t)$, $t \in T$, be an $\mathcal{H}$-valued Gaussian function on T, relative to the measure $\mathcal{P}$, with mean value 0 and correlation function $R(s,t)$. Denote by $H(\xi(t),\mathcal{P})$ the linear hull, closed in the sense of mean-square convergence, of the random variables $(x,\xi(t))$, $x \in \mathcal{H}$, $t \in T$. To any function $\varphi(\cdot) \in H(R)$ one can associate a random variable $y \in H(\xi(t),\mathcal{P})$ such that for any $x \in \mathcal{H}$ we have

$$M(x,\xi(t))y = (x,\varphi(t)), \quad t \in T;$$

this correspondence is an isomorphism between the Hilbert spaces $H(R)$ and $H(\xi(t),\mathcal{P})$; to the function $R(\cdot,t)x$ ($x \in \mathcal{H}$, $t \in T$) there corresponds the random variable $(x,\xi(t))$. Denote

$$y = \langle \varphi(t), \xi(t) \rangle.$$

Since the realizations of the random function $\xi(t)$ do not belong to $H(R)$, $\langle \varphi(t), \xi(t) \rangle$ cannot be understood literally as a scalar product; however, in concrete cases this formally written expression can be given meaning if the limiting operations over random—

···random variables entering into it is to be understood in the sense of mean-square convergence.

Denote by $H^2(\xi(t),\mathcal{P})$ the Hilbert space $H(\zeta(s,t),\mathcal{P})$, where $\zeta(s,t) = \xi(s) \otimes \xi(t) - R(s,t)$ is a random function with values in $\mathcal{H} \otimes \mathcal{H}$ and with correlation function $\Phi(s,t;u,v)$, where

$$\Phi(s,t;u,v)(x \otimes y) = R(s,u)x \otimes R(t,v)y + R(s,v)y \otimes R(t,u)x.$$

According to what was said above, there is an isomorphism between the spaces $H^2(\xi(t),\mathcal{P})$ and $H(\Phi)$, under which the random variables $(x \otimes y, \zeta(u,v))$ correspond to the $\mathcal{H} \otimes \mathcal{H}$-valued functions $\Phi(\cdot,\cdot;u,v)(x \otimes y)$, $u,v \in T$.

2. Let $\xi(t)$, $t \in T$, be an $\mathcal{H}$-valued random function on T, and let $\mathcal{B}_\xi$ be the minimal σ-algebra with respect to which all random variables $\xi(t)$, $t \in T$, are measurable; on $\mathcal{B}_\xi$ two measures $\mathcal{P}_1$ and $\mathcal{P}_2$ are given, with respect to which $\xi(t)$, $t \in T$, is a Gaussian random function with characteristics $m_i(t)$ and $R_i(s,t)$, $i = 1, 2$. Theorem 1 reduces the question of equivalence of the measures $\mathcal{P}_1$ and $\mathcal{P}_2$ and the computation of the Radon–Nikodym derivative
$$\frac{d\mathcal{P}_2}{d\mathcal{P}_1}[\xi(t)]$$
to the study of the spaces $H(R_1)$ and $H(R_2)$.

Theorem 1. *The measures $\mathcal{P}_1$ and $\mathcal{P}_2$ are either equivalent or orthogonal. $\mathcal{P}_1$ is equivalent to $\mathcal{P}_2$ if and only if the following conditions below are satisfied: 1) –3), or 1) and 4), or 2) and 5), or 6):*

1) $m_2(t) - m_1(t) \in H(R_1)$;
2) $R_2(\cdot,t)x \in H(R_1)$, and if $\langle R_2(\cdot,t)x, \varphi(\cdot)\rangle_1 = 0$* for all $x \in \mathcal{H}$, then $\varphi(t) \equiv 0$;
3) $R_2(s,t) - R_1(s,t) \in H(R_1 \otimes R_1)$;
4) $R_2(s,t) - R_1(s,t) \in H(R_1 \otimes R_2)$;
5) $R_2(s,t) - m_2(s) \otimes m_2(t) - R_1(s,t) + m_1(s) \otimes m_1(t) \in H(R_1 \otimes R_1)$;
6) $R_2(s,t) - m_2(s) \otimes m_2(t) - R_1(s,t) + m_1(s) \otimes m_1(t) \in H(R_1 \otimes R_2)$.

In the case of equivalence of the measures $\mathcal{P}_1$ and $\mathcal{P}_2$,

$$\frac{d\mathcal{P}_2}{d\mathcal{P}_1}[\xi(t)] = D \exp\left\{\frac{1}{2}\langle m_1 - m_2, m_1 - m_2\rangle_1 - \langle\tilde{\xi}, m_1 - m_2\rangle_1 + \frac{1}{2}\left\langle A(s,t), \tilde{\xi}(s) \otimes \tilde{\xi}(t) - R_1(s,t)\right\rangle_{11}\right\};$$

$$\tilde{\xi} = \xi - m_2; \qquad A(s,t) = R_1(s,t) - \langle R_1(\cdot,s), R_1(\cdot,t)\rangle_2,$$

i.e.
$$\langle f_1, f_2\rangle_1 - \langle f_1, f_2\rangle_2 = \langle A(s,t), f_1(s) \otimes f_2(t)\rangle_{11}, \qquad f_i \in H(R_1);$$
$$D = \exp\left\{\frac{1}{2}\,\mathrm{Sp}\left[\bar{A} + \ln(I - \bar{A})\right]\right\},$$

where $\bar{A}$ is the operator in $H(R_1)$ defined by the relation

$$\langle A(\cdot,t)x, f(\cdot)\rangle_1 = (x, (\bar{A}f)(t))$$

for any $x \in \mathcal{H}$, $f \in H(R_1)$. The quantities entering into the expression

$$\frac{d\mathcal{P}_2}{d\mathcal{P}_1}[\xi(t)]$$

are defined under the conditions of equivalence.*

As special cases, this theorem contains the equivalence criteria and expressions for the Radon-Nikodym derivative for measures corresponding to one-dimensional Gaussian functions (see $(2, 3, 9, 12)$), and to Gaussian measures in Hilbert spaces $(6, 7, 11, 13)$. In paragraphs 3 and 4 some applications of the results set forth are considered.

3. Let ξ be an $\mathcal{H}$-valued Gaussian random variable with zero mathematical expectation and correlation operator R. Let $R = VV^*$, where $V \in \mathcal{G}(\mathcal{H})$ (one may, for example, put $V = V^* = R^{1/2}$). Denote by V^{-1} the operator equal to $V^{-1}x = y$, if $Vy = x$, $y \in V\mathcal{H}$, and equal to $V^{-1}x = 0$, if $Vx = 0$. An element $m \in H(R)$ if and only if $m = Vx$, $x \in \mathcal{H}$;

$$\langle m_1, m_2 \rangle = (V^{-1}m_1, V^{-1}m_2).$$

Formally,

$$\langle m, \xi \rangle = (V^{-1}m, V^{-1}\xi).$$

This expression can be given, for example, the following meaning. Let $e_1, e_2, \ldots$ be a sequence of eigenvectors of the operator R, corresponding to the nonzero eigenvalues $\lambda_1, \lambda_2, \ldots$. Then

$$\langle m_1, m_2 \rangle = \sum_i \frac{1}{\lambda_i}(m_1, e_i)(m_2, e_i), \qquad \langle m, \xi \rangle = \sum_i \frac{1}{\lambda_i}(m, e_i)(\xi, e_i)$$

(the series converges in mean and almost everywhere). The space $H(R_1 \otimes R_2)$, $(R_1, R_2 \in K(\mathcal{H}))$, consists of operators $K \in \mathcal{G}(\mathcal{H})$ of the form $K = V_1BV_2^*$, where $B \in \mathcal{G}(\mathcal{H})$,

$$* \quad \langle \cdot, \cdot \rangle_i = \langle \cdot, \cdot \rangle_{H(R_i)}, \quad \langle \cdot, \cdot \rangle_{i,j} = \langle \cdot, \cdot \rangle_{H(R_i \otimes R_j)}.$$

$R_i = V_{iV}i^*,\ i = 1, 2;\ \langle K_1, K_2 \rangle_{12} = (B_1, B_2)$, and if $R_1 = R_2 = R$, then

$$\langle K_1, K_2 \rangle = \sum \frac{1}{\lambda_i^2 \lambda_j}(K_1e_i, e_j)(K_2e_i, e_j),$$

$$\langle K, \xi \otimes \xi - R \rangle = \sum \left(\frac{1}{\lambda_i \lambda_j} - \delta_{ij} \right) (Ke_i, e_j)(\xi, e_i)(\xi, e_j)$$

(the series converges in mean and almost everywhere).

Theorem 2. Let ξ be an $\mathcal{H}$-valued Gaussian random variable with respect to the measures $\mathcal{P}_1$ and $\mathcal{P}_2$; let m_i and R_i $(i = 1, 2)$ be its means and correlation operators, and let $R_i = V_{iV}i^*$. The measures $\mathcal{P}_1$ and $\mathcal{P}_2$ are equivalent on $\mathcal{B}_\xi$ if and only if the following conditions are satisfied: 1) $m_2 - m_1 = V_1x,\ x \in \mathcal{H}$; 2) $\overline{R_1\mathcal{H}} = \overline{R_2\mathcal{H}}$; 3) $R_2 - R_1 = V_1BV_1^*,\ B \in \mathfrak{G}(\mathcal{H})$, or condition 1) and condition 4) $R_2 - R_1 = V_1CV_2^* = V_2C^*V_1^*,\ C \in \mathfrak{G}(\mathcal{H})$, or condition 2) and condition 5) $R_2 - m_2 \otimes m_2 - R_1 + m_1 \otimes m_1 = V_1DV_1^*,\ D \in \mathfrak{G}(\mathcal{H})$, or condition 6)

$R_2 - m_2 \otimes m_2 - R_1 + m_1 \otimes m_1 = V_1 E V_2^*,\ E \in \mathfrak{G}(\mathcal{H})$. The computation of $\dfrac{d\mathcal{P}_2}{d\mathcal{P}_1}[\xi]$ is carried out using the formula given in Theorem 1.

The operator

$$A = V_1[I - (V_2^{-1}V_1)^* V_2^{-1} V_1] V_1^*, \qquad \bar{A} = I - V_1 (V_2^{-1} V_1)^* V_2^{-1}$$

(cf. ([6],[7],[11],[13])).

4. Let $T = [a, b]$, $-\infty < a < b < \infty$, $\mathcal{H} = R^n$, and let $\Psi(t)$ and $\Phi(t)$ be operator functions, where $\Psi^{-1}(t)$ and $\Phi^{-1}(t)$ exist.

Theorem 3. The operator function

$$R(s,t) = \Psi(s)\Phi^*(t),\ s \le t; \qquad R(s,t) = \Phi(s)\Psi^*(t),\ s \ge t, \tag{1}$$

is positive definite if and only if:

1) $R(t,t) = \Psi(t)\Phi^*(t) \in K_+(\mathcal{H})$ for every $t \in [a, b]$;
2) $\Lambda(t) = \Phi^{-1}(t)\Psi(t) \in K_+(\mathcal{H}),\ t \in [a, b]$;
3) $\Lambda(t) - \Lambda(s) \in K_+(\mathcal{H})$ for $t > s,\ s, t \in [a, b]$.

Theorem 4. A nondegenerate mean-continuous $\mathcal{H}$-valued Gaussian random process $\xi(t)$, $t \in [a, b]$, is Markovian if and only if its correlation function is representable in the form (1).

From the properties of $\Lambda(t)$ it follows that, for almost all $t \in [a, b]$, there exists a derivative $\Lambda'(t) \in K_+(\mathcal{H})$. Below, for convenience, it is assumed that all operators $\Lambda'(t)$ are nondegenerate. Introduce the functions $M(t)$ and the operator V:

$$\Lambda'(t) = M(t)M^*(t), \qquad \Lambda(a) = VV^*$$

(for example, one may take $M(t) = M^*(t) = [\Lambda'(t)]^{1/2}$, $V = V^* = [\Lambda(a)]^{1/2}$).

Theorem 5. Let $\xi(t)$, $t \in [a, b]$, be an R^n-valued process with correlation function of the form (1); then

$$w(t) = \int_a^t M^{-1}(t)\, d[\Phi^{-1}(t)\xi(t)] + V^{-1}\Phi^{-1}(a)\xi(a)$$

is an R^n-valued Wiener process with correlation function $I \min(s, t)$. Conversely,

$$\xi(t) = \Phi(t) \int_a^t M(t)\, dw(t) + \xi(a).$$

Assume further, for convenience, that $\xi(a) = 0$. An R^n-valued function $m(t) \in H(R)$ if and only if the function $\Phi^{-1}(t)m(t)$ is absolutely continuous and

$$f(t) = M^{-1}(t)(\Phi^{-1}(t)m(t))' \in L_2.$$

If $m_1(t), m_2(t) \in H(R)$, then

$$\langle m_1, m_2\rangle = \int_a^b (f_1(t), f_2(t))\,dt; \qquad \langle m, \xi\rangle = \int_a^b (M^{-1}(t)(\Phi^{-1}(t)m(t))', M^{-1}(t)\,d(\Phi^{-1}(t)\xi(t))).$$

Let R_1 and R_2 be two kernels of the form (1). The space $H(R_1 \otimes R_2)$ consists of operator functions $m(s,t)$ such that $\Phi_1^{-1}(s)m(s,t)\Phi_2^{*-1}(t)$ is an absolutely continuous function and

$$f(s,t) = M_1^{-1}(s)\frac{\partial^2}{\partial s\,\partial t}(\Phi_1^{-1}(s)m(s,t)\Phi_2^{*-1}(t))M_2^{*-1}(t) \in L_2;$$

$$\langle m_1, m_2\rangle_{1,2} = \int_a^b\int_a^b \mathrm{Sp}[f_1(s,t)f_2^*(s,t)]\,ds\,dt.$$

$H^2(\xi(t), \mathcal{P})$ consists of random variables of the form

$$\langle m(s,t), \xi(s)\times\xi(t) - R(s,t)\rangle =$$

$$= \int_a^b\int_a^b \mathrm{Sp}\left\{[\Lambda'(s)]^{-1}\frac{\partial^2}{\partial s\,\partial t}\left(\Phi^{-1}(s)m(s,t)\Phi^{*-1}(t)\right)[\Lambda'(t)]^{-1}\,d_s d_t\left(\Phi^{-1}(t)\times\right.\right.$$
$$\left.\left.\times\,[\xi(s)\otimes\xi(t)]\Phi^{*-1}(s)\right)\right\},$$

where the double stochastic integral is understood in the sense of K. Itô; it is defined for any function $m(s,t) \in H(R\otimes R)$ (cf. $(2^3, 5)$).

Let $\xi(t)$, $t \in [a,b]$, be a continuous in the mean nondegenerate n-dimensional Gaussian process with characteristics $m_i(t)$ and $R_i(s,t)$ with respect to the measures $\mathcal{P}_i$, $i = 1,2$, on $\mathfrak{B}_\xi$. Denote $K_{12}(t) = \Phi_1^{-1}(t)\Phi_2(t)$ and $K_{21}(t) = K_{12}^{-1}(t) = \Phi_2^{-1}(t)\Phi_1(t)$.

Theorem 6. The measures $\mathcal{P}_1$ and $\mathcal{P}_2$ are equivalent if and only if:

1) the vector function $\Phi_1^{-1}(t)(m_2(t) - m_1(t))$ is absolutely continuous and

$$M_1^{-1}(t)[\Phi_1^{-1}(t)(m_2(t) - m_1(t))]' \in L_2;$$

2) $\Lambda_1'(t)K_{21}'(t) = K_{12}(t)\Lambda_2'(t)$ almost everywhere on $[a, b]$;

3) $K_{12}'(t)$ and $K_{21}'(t)$ exist almost everywhere;

4) $f(s, t) \in L_2$, where $f(s, t) = [M_2^{-1}(t)K_{21}'(t)M_1(s)]^*$ for $s \le t$, and

$$f(s, t) = M_1^{-1}(s)K_{12}'(t)M_2(t)$$

for $s > t$.

The computation of

$$\frac{d\mathcal{P}_2}{d\mathcal{P}_1}[\xi(t)]$$

is carried out by the formula of Theorem 1. The operator function

$$A(s, t) = \Phi_1(s)\widetilde{A}(s, t)\Phi_1^*(t),$$

where

$$\widetilde{A}(s, t) = \int_a^s K_{21}(u)K_{12}'(u)\Lambda_1(\min(t, u))\, du + \int_a^t \Lambda_1(\min(s, u))K_{12}^{*\prime}(u)\times$$

$$\times K_{21}^*(u)\, du - \int_a^b \Lambda_1(\min(s, u))K_{12}^{*\prime}(u)K_{21}^*(u)(\Lambda_1'(u))^{-1}K_{21}(u)K_{12}'(u)\times$$

$$\times\Lambda_1(\min(t, u))\, du; \qquad \ln D = \int_a^b \int_t^b \mathrm{Sp}\,[K_{12}^*(u)K_{21}^*(u)(\Lambda_1'(u))^{-1}K_{21}(u)K_{12}'(u)]\, du.$$

Let us consider, in particular, a random process $\xi(t)$ which, with respect to the measures $\mathcal{P}_i$, $i = 1, 2$, on $\mathfrak{B}_\xi$, is a stationary Gaussian Markov process with $m_i(t) \equiv 0$;

$$R_i(s, t) = C_i \exp\{|t - s|Q_i\}C_i,$$

where $-(Q_i + Q_i^*)$ and C_i are positive definite operators; here

$$\Psi_i(t) = C_i \exp\{-tQ_i\}, \qquad \Phi_i(t) = C_i \exp\{tQ_i^*\}, \qquad \Lambda_i(t) = \exp\{-tQ_i\}\exp\{-tQ_i^*\}.$$

It is not hard to verify that $\mathcal{P}_1$ and $\mathcal{P}_2$ are equivalent if and only if

$$C_1(Q_1 + Q_1^*)C_1 = C_2(Q_2 + Q_2^*)C_2.$$

The measures $\mathcal{P}_1$ and $\mathcal{P}_2$ corresponding to two Gaussian processes with

$$m_i(t) \equiv 0, \qquad R_1(s, t) = C\exp\{|t - s|Q\}C, \qquad R_2(s, t) = B\min(s, t),$$

are equivalent if and only if

$$B = -C(Q + Q^*)C.$$

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Note: Figure translations are in progress. See original paper for figures.