MAPPINGS OF METRIC SPACES

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Abstract

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One of the main directions in general topology is the elucidation of connections established between classes of spaces by means of mappings of various types. The basic class of topological spaces is that of metrizable spaces. Therefore one of the tasks is to describe, in intrinsic terms, partitions of metric spaces under various restrictions on the elements of the partition. In the works ([2,3,8]) the connection was clarified between the class of metric spaces and the class of spaces with a refining sequence of coverings. An important role in this was played by the definition of the notion of a π-mapping, proposed by V. I. Ponomarev. A mapping $f : X \to Y$ of a metric space X onto a topological space Y is called a π-mapping if, for any point $x \in Y$ and any neighborhood Ox of it, $\rho(f^{-1}x, X \setminus f^{-1}Ox) > 0$. All spaces considered below are assumed to be T_1-spaces, and all mappings continuous.

Definition 1. A system of coverings (not necessarily open)

$$\{\gamma_n = \{H_\alpha \mid \alpha \in A_n\};\ n = 1, 2, ...\}$$

of a space X is called quasi-refining if a set P is closed in X if and only if, for each point $x \in X \setminus P$, there exists a natural number n such that

$$\left(\bigcap_{k=1}^{n} \gamma_k x\right) \cap P = \emptyset, \quad \text{where } \gamma_k x = U\{H_\alpha \in \gamma_k \mid H_\alpha \ni x\}.$$

Theorem 1. *The following conditions are equivalent: 1) The space X has a quasi-refining sequence of coverings. (2) The space X is a quotient π-image of some metric space.*

Proof. (2) → (1). Let $f : Z \to X$ be a quotient π-mapping of the metric space Z onto the space X. Put

$$\gamma_n = \{fU_\alpha \mid U_\alpha \subseteq Z \text{ and } \operatorname{diam} U_\alpha < 1/n\}.$$

We shall prove that the system $\{\gamma_n \mid n = 1, 2, ...\}$ quasi-refines. Let the set P be closed in X and let $x \in X \setminus P$. Then, by the condition, we have

$$\rho(f^{-1}x, f^{-1}P) > 1/n > 0.$$

Therefore $\gamma_n x \cap P \neq \emptyset$. Let, further, the set P be such that for any point $x \in X$ there is a natural number n such that $\gamma_n x \cap P = \emptyset$.

We shall prove that the set $f^{-1}P$ is closed in Z, i.e. the set P is closed in X. Indeed, if $z \notin f^{-1}P$, then $f^{-1}fz \cap f^{-1}P = \emptyset$, and for some n we have $\gamma_n fz \cap P = \emptyset$; consequently,

$$O(z, 1/n) \cap f^{-1}P = \emptyset.$$

This proves the quasi-refinement of the coverings $\{\gamma_n \mid n = 1, 2, \ldots\}$.

(1) → (2). Let

$$\{\gamma_n = \{H_\alpha \mid \alpha \in A_n\},\ n = 1, 2, \ldots\}$$

be a sequence of quasi-refining coverings of the space X. Put

$$Z = \prod_{n=1}^{\infty} A_n,$$

where A_n is a discrete space, i.e. for any $a_1, a_2 \in A_n$ we have

$$\rho_n(a_1, a_2) = 1, \quad \text{if } a_1 \neq a_2.$$

The metric on the space Z is the following: if

$$(\alpha_1, \ldots, \alpha_n, \ldots),\ (\beta_1, \ldots, \beta_n, \ldots) \in Z,$$

then

$$\rho((\alpha_1, \ldots, \alpha_n, \ldots), (\beta_1, \ldots, \beta_n, \ldots)) = \left[\sum_{n=1}^{\infty} \frac{1}{2^n} \rho_n^2(\alpha_n, \beta_n)\right]^{1/2}.$$

A point $(\alpha_1, \ldots, \alpha_n, \ldots) \in Z$ will be called distinguished if

$$\bigcap_{n=1}^{\infty} H_{\alpha_n} \neq \emptyset.$$

Let us denote the set of marked points by Z_0. Put $f : Z_0 \to X$, where

$$f(\alpha_1, \ldots, \alpha_n, \ldots) = \bigcap_{n=1}^{\infty} H_{\alpha_n}.$$

It is clear that $\bigcap_{n=1}^{\infty} H_{\alpha_n}$ consists of only one point. Denote

$$A_{nx} = \{\alpha \mid \alpha \in A_n \text{ and } H_\alpha \ni x\}.$$

It is easily proved that the mapping f is continuous and that for every point $x \in X$ the formula

$$f^{-1}x = \prod_{n=1}^{\infty} A_{nx} \tag{*}$$

holds.

We shall prove that f is a π-mapping. Let the point x and its neighborhood Ox be arbitrary. There exists a natural number n such that

$$\bigcap_{k=1}^{n} \gamma_k x \subseteq Ox.$$

Then, if $(\alpha_1, \dots, \alpha_n, \dots) \in f^{-1}x$ and $(\beta_1, \dots, \beta_n, \dots) \in Z_0 \setminus f^{-1}Ox$, then $\alpha_k \neq \beta_k$, where $k \leq n$, and therefore

$$\rho((\alpha_1, \dots, \alpha_n, \dots), (\beta_1, \dots, \beta_n, \dots)) \geq 1/n.$$

Consequently,

$$\rho(f^{-1}x,\ Z_0 \setminus f^{-1}Ox) \geq 1/n > 0.$$

Now we shall prove that the mapping f is factor, i.e., we shall prove: if a set P is not closed in X, then the set $f^{-1}P$ is not closed in X. Indeed, if the set P is not closed in X, then there exist a point $x \in X \setminus P$ and sets $H_{\alpha_n} \in \gamma_n$ such that

$$x \in H_{\alpha_p} \quad \text{and} \quad \left(\bigcap_{k=1}^{n} H_{\alpha_k} \right) \cap P \neq \emptyset$$

for every natural number n. It is clear that

$$(\alpha_1, \dots, \alpha_n, \dots) \in f^{-1}x.$$

We shall prove that

$$(\alpha_1, \dots, \alpha_n, \dots) \in [f^{-1}P]_{Z_0}.$$

From the condition

$$\left(\bigcap_{k=1}^{n} H_{\alpha_k} \right) \cap P \neq \emptyset$$

it follows that there exists a point

$$z_m = (\beta_1, \dots, \beta_n, \dots) \in f^{-1}P,$$

where $\beta_k = \alpha_k$ for $k = 1, 2, \dots, m$. Therefore the sequence $\{z_m \mid m = 1, 2, \dots\}$ converges to the point $(\alpha_1, \dots, \alpha_n, \dots)$, and this proves Theorem 1.

From formula (*) one proves, by analogous arguments,

Theorem 2. *The following conditions are equivalent: (1) The space X has a quasi-refining sequence of point-finite covers. (2) The space X is a factor bicompact image of some metric space.*

The theorem from (6) and Theorem 1 allow us to obtain the following theorem:

Theorem 3. *A paracompact p-space with a quasi-refining sequence of covers is metrizable.*

Definition 2. A mapping $f : X \to Y$ of a metric space X onto a space Y is called a T_1-mapping*, if the distance between any two complete inverse images of points is positive.

Definition 3. A sequence of open covers

$$\{\gamma_n = \{U_\alpha \mid \alpha \in A_n\},\ n = 1, 2, ...\}$$

is called weakly refining if the following conditions are satisfied: (a) for every point $x \in X$ we have

$$\bigcap_{n=1}^{\infty} \gamma_n x = x;$$

(b) for every point $x \in X$ there exists a sequence U_{α_n}, where $\alpha_n \in A_n$, forming a base at the point x.

The following theorem explains which spaces have a weakly refining sequence of covers.

Theorem 4. *Let a space X with the first axiom of countability satisfy one of the following conditions: (1) X has a countable net and is regular. (2) The space X is regular and has a σ-discrete net. (3) $X \times X$ is perfectly normal. (4) The space X is condensed onto some metric space. (5) The diagonal $\{(x, x) \mid x \in X\}$ has type G_δ in $X \times X$. Then X has a weakly refining sequence of covers.*

Proof. It is easily proved that conditions (1)–(4) reduce to condition (5). Let the diagonal

$$D = \{(x, x) \mid x \in X\}$$

be a countable inter–

by the intersection of a countable number of sets open in $X \times X$, i.e.

$$D = \bigcap_{n=1}^{\infty} U_n,$$

where U_n is open in $X \times X$ for every natural number n.

Put

$$\omega_n = \{V_\alpha \mid V_\alpha \text{ is open in } X \text{ and } V_\alpha \times V_\alpha \subseteq U_n\}.$$

Since

$$D = \bigcap_{n=1}^{\infty} U_n,$$

for every point $x \in X$ we have

$$\bigcap_{n=1}^{\infty} \omega_n x = x.$$

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* T_1-mappings were introduced in the paper (7).

Let

$$\Omega = \{\Gamma_\beta \mid \beta \in B\}$$

be some open base of the space X. Put

$$\gamma_n = \omega_n \cap \Omega = \{U_\alpha \cap \Gamma_\beta \mid U_\alpha \in \omega_n,\ \Gamma_\beta \in \Omega\}.$$

Since the cover γ_n is inscribed in the cover ω_n, for every point $x \in X$ we have

$$\bigcap_{n=1}^{\infty} \gamma_n x = x.$$

Now, from the condition that Ω forms a base in the space X, and from the fact that X satisfies the first axiom of countability, it follows that the sequence of covers $\{\gamma_n \mid n = 1, 2, \ldots\}$ is weakly refining.

Remark. Condition (5) is necessary and sufficient for a space with the first axiom of countability to possess a weakly refining sequence of covers.

Theorem 5. *The following conditions are equivalent:*

(1) The space X possesses a weakly refining sequence of open covers.

(2) The space X is an open T_1-image of some metric space.

Proof. It is easy to show that condition (2) implies condition (1). Let

$$\gamma_n = \{U_\alpha \mid \alpha \in A_n\}$$

be a sequence of weakly refining covers. Put

$$Z = \prod_{n=1}^{\infty} A_n,$$

where A_n is discrete. A point

$$(a_1, a_2, \ldots, a_n, \ldots) \in Z$$

will be called marked if the sets U_{α_n} form a base at some point of the space X. Denote by Z_0 the set of all marked points and put $f : Z_0 \to X$, where

$$f(a_1, \ldots, a_n, \ldots) = \bigcap_{n=1}^{\infty} U_{\alpha_n}.$$

The mapping f is continuous and $fZ_0 = X$. The openness of the mapping f follows from the formula

$$fO(a_1, \ldots, a_n) = \bigcap_{k=1}^{n} U_{\alpha_k},$$

where

$$O(a_1, \ldots, a_m) = \{(\beta_1, \ldots, \beta_n, \ldots) \mid \beta_k = \alpha_k \text{ for } k \leq m\},$$

and from the fact that sets of the form $O(a_1, \dots, a_m)$ form a base in the space Z_0.

Let us prove that f is a T_1-mapping. Let $x, y \in X$ and $x \neq y$. By hypothesis there exists a natural number n such that $\gamma_n x \not\ni y$. Therefore, if

$$(a_1, \dots, \alpha_n, \dots) \in f^{-1}x$$

and

$$(\beta_1, \dots, \beta_n, \dots) \in f^{-1}y,$$

then $\alpha_n \neq \beta_n$ and

$$\rho((a_1, \dots, \alpha_n, \dots), (\beta_1, \dots, \beta_n, \dots)) \geq 1/n.$$

Consequently,

$$\rho(f^{-1}x, f^{-1}y) \geq 1/n.$$

Theorem 5 is proved.

From Theorem 5 and the proposition from (5) we obtain

Theorem 6. *Let $f : X \to Y$ be an open T_1-mapping of a metric space X onto a paracompact p-space Y. Then Y is metrizable.*

We note that Theorem 6 is false in the class of paracompact spaces, since every regular space with a countable net and the first axiom of countability, by Theorems 4 and 5, is an open T_1-image of some metric space, but among them there are also nonmetrizable ones.

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Note: Figure translations are in progress. See original paper for figures.