ON AN INVARIANT MEASURE FOR A \(Y\)-FLOW ON A THREE-DIMENSIONAL MANIFOLD

MATHEMATICS

1969

SovietRxiv

Abstract

Full Text

UDC 519.217

MATHEMATICS

M. E. RATNER

ON AN INVARIANT MEASURE FOR A Y-FLOW ON A THREE-DIMENSIONAL MANIFOLD

(Presented by Academician A. N. Kolmogorov, 28 X 1968)

In [1] it is proved that if T is a Y-diffeomorphism of class C^2 of the two-dimensional torus M, having everywhere dense leaves of the transversal foliations, then in M there exists a measure μ, positive on every open set, invariant with respect to T, and T, as an automorphism of the space with measure (M, μ), is a K-automorphism. Moreover, for every measurable partition ξ of the space M, whose elements C_ξ are intervals of the contracting foliation, the conditional measure $\mu(\cdot \mid C_\xi)$, induced by μ, is equivalent on almost every C_ξ to the normalized Riemannian volume on C_ξ. The listed properties determine the measure μ uniquely. The proof in [1] is based on the existence of a Markov partition. In the present paper an analogous assertion is proved for a Y-flow on a three-dimensional manifold.

1. Markov partition. Let W be a three-dimensional compact Riemannian manifold of class C^∞; $\{T^t\}$ a Y-flow of class C^2 on W (see [2]); Γ_c (Γ_p) its contracting (expanding) foliation; G_c (G_p) the foliation into leaves

$$G_c = \bigcup_{t=-\infty}^{\infty} T^t \Gamma_c \quad \left(G_p = \bigcup_{t=-\infty}^{\infty} T^t \Gamma_p \right).$$

It is assumed that all leaves of the foliations Γ_c and Γ_p are everywhere dense in W.

Let the points w_0 and w_1 lie on one small interval $\gamma_c(w_0)$ of the leaf $\Gamma_c(w_0)$; let M_{w_0} and M_{w_1} be neighborhoods of the points w_0 and w_1 on $G_p(w_0)$ and $G_p(w_1)$ such that for $w \in M_{w_0}$ there is a $w' \in M_{w_1}$, lying on a small interval $\gamma_c(w)$, and the mapping $\pi_{w_0,w_1} : w \to w'$ is a homeomorphism of M_{w_0} onto M_{w_1}. Let $\gamma_p \in M_{w_0}$ be an interval of the leaf $\Gamma_p(w_0)$; let $\tau(w)$ be a continuous nonnegative function on γ_p, not identically equal to zero.

Every set

$$\Pi = \bigcup_{w \in \gamma_p} \gamma_c(w) \quad \left(V = \bigcup_{w \in \gamma_p} \bigcup_{\tau=0}^{\tau(w)} T^\tau \gamma_c(w) \right)$$

will be called a Γ_c-regular parallelogram (parallelepiped), and the lengths of the intervals γ_p and $\gamma_c(w_0)$ the dimensions of the parallelogram Π. The contracting Δ_c (expanding Δ_p) boundary of V will be called

$$\Delta_c(V) = \bigcup_{y\in\partial\gamma_p} \bigcup_{\tau=0}^{\tau(y)} T^\tau \gamma_c(y) \quad \left(\Delta_p(V) = \bigcup_{w\in\gamma_p} \bigcup_{\tau=0}^{\tau(w)} T^\tau \left(w \cup \pi_{w_0,w_1}(w)\right)\right).$$

We shall call Π the upper face of V. If β is a collection of parallelepipeds $\{V_i\}$, then $\Delta_c(\beta) = \bigcup_i \Delta_c(V_i)$ and $\Delta_p(\beta) = \bigcup_i \Delta_p(V_i)$.

Analogously to [1], we introduce the following definition.

Definition. A finite partition α into nonintersecting Γ_c-regular parallelepipeds $\{V_i\}$ is called **Markov** if for all $t \geq 0$, $T^t\Delta_c(\alpha) \subset \Delta_c(\alpha)$, and for all $t \leq 0$, $T^t\Delta_p(\alpha) \subset \Delta_p(\alpha)$.

Theorem 1. *For every $\varepsilon > 0$ there exists a Markov partition into Γ_c-regular parallelepipeds whose upper-face dimensions do not exceed ε.*

2. Special representation.

Let a be a Markov partition and let M be the set-theoretic union of the upper faces $\{\Pi_i\}$ of the parallelepipeds $\{V_i\}$. Let ξ_c (ξ_p) be the partition of M into intervals C_{ξ_c} (C_{ξ_p}), belonging to $\{\Pi_i \cap \Gamma_c\}$ ($\{\Pi_i \cap C_p\}$). The intervals C_{ξ_p} are not, generally speaking, intervals of leaves Γ_p. However, through each point $y \in C_\xi$ one can draw an interval $\gamma C_{\xi_p}(y)$ of the leaf $\Gamma_p(y)$ such that there exists a continuous function $q(z)$, for $z \in \gamma C_{\xi_p}(y)$, having the properties: 1) $q(\gamma C_{\xi_p}(y) \cap C'_{\xi_p}) = 0$; (2) $T^{q(z)}z = C_{\xi_p}$; 3) the mapping $\psi : z \to T^{q(z)}z$ is a homeomorphism of $\gamma C_{\xi_p}(y)$ onto C_{ξ_p}.

For $x_0 \in \Pi_i$, consider $C_{\xi_c}(x_0)$ and $C_{\xi_p}(x_0)$. Let u_1 (u_2) be a smooth coordinate on the leaf $\Gamma_c(x_0)$ ($\Gamma_p(x_0)$) such that $u_1(x_0) = 0$ ($u_2(x_0) = 0$). Consider on $C_{\xi_c}(x_0)$ the coordinate u_1, and on $C_{\xi_p}(x_0)$ the coordinate u_2, transferred from $\gamma C_{\xi_p}(x_0)$ by means of the mapping ψ. For every point $w \in \Pi$ there are points $z_1 \in C_{\xi_c}(x_0)$ with coordinate u_1 and $z_2 \in C_{\xi_p}(x_0)$ with coordinate u_2 such that $w = C_{\xi_c}(z_2) \cap C_{\xi_p}(z_1)$. Assign to w the coordinates (u_1, u_2). Then Π_i becomes a smooth manifold with boundary, and M will consist of a finite number of such manifolds, generally not connected with one another. Consider on Π_i the mapping $\chi : w \to (u_1, u_2)$. It is a homeomorphism of Π_i onto a rectangle P_i of the Euclidean plane (u_1, u_2). We introduce on Π_i Riemannian lengths transferred from P_i by means of the mapping χ^{-1}.

From the absolute continuity of the foliations G_p and G_c (see (2)) it follows that the Riemannian length on C_{ξ_c} in Π_i is equivalent to the Riemannian length on C_{ξ_c} induced by the Riemannian metric in W, while the Riemannian length on C_{ξ_p} is equivalent to the Riemannian length on C_{ξ_p}, transferred from $\gamma C_{\xi_p}(x)$, $x \in C_{\xi_p}$, by means of the mapping ψ.

Through $x \in M$ draw an interval of length $l(x)$ of the trajectory of the flow $\{T^t\}$ in the positive direction up to the first intersection with M at the point x'. The mapping $U : x \to x'$ is a one-to-one mapping of M onto itself, and the flow $\{T^t\}$ is represented as a special flow (see (3)) over (M, U) with function $l(x)$. The mapping U on M is similar in its properties to a Y-diffeomorphism: it uniformly exponentially contracts (expands) the intervals $C_{\xi_c}(C_{\xi_p})$. The mapping U cannot be called a Y-diffeomorphism, since M is not a connected closed manifold and U has, generally speaking, discontinuity points.

The partition $\{\Pi_i\}$ is Markov for U, and the partition ξ_c is increasing. The function $l(x)$ is constant on C_{ξ_c}. Using the Hölder condition for the foliation Γ_c (see (4)), one can prove that $l(y)$, $y \in C_{\xi_p}$, satisfies a Hölder condition of positive order on each component of its continuity.

Using (4) and the absolute continuity of the foliation G_p, by the methods of the work [1] one can prove the following theorem.

Theorem 2. *In the space M there exists a measure m, positive on every open set, invariant with respect to U, and U, as an automorphism of the space M with measure m, is a K-automorphism with K-partition ξ_c. The conditional measure $m(\cdot \mid C_{\xi_c})$, induced by m on almost every C_{ξ_c}, is equivalent to the normalized Riemannian volume on C_{ξ_c}. These properties determine the measure m uniquely.*

3. Invariant measure.

Using the special representation of the flow $\{T^t\}$, consider in W the normalized measure

$$d\mu = \frac{1}{\bar{l}}(dm \times dt).$$

$$\bar{l} = \int_M l(x)\, dm$$

and the partition η_c, consisting of the segments

$$\{T^u C_{\xi_c}(x),\ 0 \le u < l(x),\ x \in M\}.$$

From the invariance of the measure m with respect to U, the invariance of the measure μ with respect to the flow $\{T^t\}$ follows easily. The partition η_c has the following properties: 1) $T^{t_1}\eta_c < T^{t_2}\eta_c$ for $t_1 < t_2$;

2)

$$\prod_{t=-\infty}^{\infty} T^t \eta_c = \varepsilon,$$

where ε is the partition into individual points; 3)

$$\bigcap_{t=-\infty}^{\infty} T^t \eta_c = \nu_Z,$$

where ν_Z is the measurable envelope of the partition Z into full layers of the foliation Γ_c.

It follows from (5) that if the foliations Γ_c and Γ_p form a nonintegrable pair, then $\nu_Z = \nu$, where ν is the trivial partition whose only element is the whole space.

Thus the following theorem has been proved:

Theorem 3. In the space W there exists a measure μ, positive on every open set, invariant with respect to the flow $\{T^t\}$, and such that the conditional measure $\mu(\cdot/C_{\eta_c})$, induced by μ on almost every C_{η_c}, is equivalent to the normalized Riemannian volume on C_{η_c}. If the foliations Γ_c and Γ_p form a nonintegrable pair in the sense of (5), then the flow $\{T^t\}$ in (W, μ) is a K-flow with K-partition η_c. The measure μ is uniquely determined by these properties.

For the geodesic flow on a compact manifold of negative curvature, the measure μ coincides with the invariant Riemannian volume.

In conclusion, the author expresses gratitude to Ya. G. Sinai, under whose supervision the present work was carried out.

Moscow State University
named after M. V. Lomonosov

Received
20 X 1968

REFERENCES

1. Ya. G. Sinai, *Functional Analysis and Its Applications*, **2**, no. 1 (1968).

2. D. V. Anosov, *Proceedings of the V. A. Steklov Mathematical Institute, Academy of Sciences of the USSR*, **40** (1967).

3. V. A. Rokhlin, *Uspekhi Mat. Nauk*, **4**, no. 2 (1949).

4. D. V. Anosov, *Mathematical Notes*, **2**, no. 5 (1967).

5. Ya. G. Sinai, *Izv. Academy of Sciences of the USSR, Ser. Math.*, **30**, no. 1 (1966).

Note: Figure translations are in progress. See original paper for figures.