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MATHEMATICS

1969

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Abstract

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UDC 513.836

MATHEMATICS

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ON THE COHOMOLOGY OF ALGEBRAS WITH IDENTITY

(Presented by Academician P. S. Aleksandrov on 29 IV 1968)

Let Λ be an associative-commutative unital K-algebra with identity over an associative-commutative ring with identity K. In the paper [1], Yates proposed a method for defining the cohomology $H^p(\Lambda)$ of the algebra Λ. For each $p \geq 0$, the K-module $H^p(\Lambda)$ is a covariant functor on the category of associative-commutative unital K-algebras with identity (the ring K has an identity), whose morphisms are homomorphisms of K-algebras preserving the identity. Yates showed that the modules $H^p(\Lambda)$ coincide with the Alexander-Čech cohomology groups $H^p(X; R)$, if for Λ one takes the algebra of all continuous real-valued functions on a bicompact space X.

Let X_Λ be the prime spectrum of the algebra Λ in the Zariski topology. In this note, for any subset $T \subset X_\Lambda$, cohomology groups $H^p(\Lambda; T)$ are defined; there is an exact cohomology sequence of the pair (X_Λ, T) (see Theorem 1). In the case when T is the set of all maximal ideals, the K-modules $H^p(\Lambda; T)$ coincide with the Yates modules $H^p(\Lambda)$. For $T = X_\Lambda$, the modules $H^p(\Lambda; T)$ are covariant functors in the category under consideration. For the pair (X_Λ, T), on

$$H^*(\Lambda; X_\Lambda, T) = \bigoplus_{p=0}^{\infty} H^p(\Lambda; X_\Lambda, T)$$

there is defined the structure of an associative skew-commutative K-algebra (see Theorem 2). In the case when Λ is the algebra of all continuous real-valued functions on a paracompact space X, and T is the set of all maximal ideals, each of which consists of all functions vanishing at some point, the algebra $H^*(\Lambda; T)$ is isomorphic to the Alexander-Čech cohomology algebra $H^*(X; R)$ (see Theorem 3).

§ 1. Put $F^0 = \Lambda;\ F^q = F^{q-1} \otimes_K \Lambda$. For a homogeneous element $x_0 \otimes \cdots \otimes x_q \in F^q$, put

$$\mu^q(x_0 \otimes \cdots \otimes x_q) = x_0 \cdots x_q \in \Lambda.$$

It is clear that $\mu^q : F^q \to \Lambda$ is an epimorphism of K-algebras. Define homomorphisms of algebras

$$d_i^q : F^q \to F^{q+1}, \qquad i = 0, 1, \dots, q+1,$$

by the formula

$$d_i^q(x_0 \otimes \cdots \otimes x_q) = x_0 \otimes \cdots \otimes x_{i-1} \otimes 1 \otimes x_i \otimes \cdots \otimes x_q$$

for $i = 1, 2, \dots, q$, and

$$d_0^q(x_0 \otimes \cdots \otimes x_q) = 1 \otimes x_0 \otimes \cdots \otimes x_q, \qquad d_{q+1}^q(x_0 \otimes \cdots \otimes x_q) = x_0 \otimes \cdots \otimes x_q \otimes 1$$

for $i = 0, q+1$.

Put

$$d^q = \sum_{i=0}^{q+1} (-1)^i d_i^q.$$

Obviously, d^q are homomorphisms of K-modules,

$$d^q : F^q \to F^{q+1}.$$

It is not hard to verify that $d^{q+1} d^q = 0$. Thus the set $\{F^q, d^q\}$ is transformed into a cochain complex F of K-modules. Let $\alpha \in X_\Lambda$ be a prime ideal in Λ.

Definition 1. We shall say that $x \in F^q$ **vanishes near** α if there exists such a $y \in F^q$ that: 1) $\mu^q y \notin \alpha$; 2) $xy = 0$. Such an element will be called an **annihilator** of x.

We shall say that $x \in F^q$ **vanishes near the set** $T \subset X_\Lambda$ if x vanishes near every $\alpha \in T$.

Proposition 1. *If $x \in F^q$ vanishes near $a \in X_\Lambda$, then x vanishes near some neighborhood of a.*

Indeed, in this case there exists an element $y \in F^q$ satisfying conditions 1), 2) of Definition 1. Consider $U = \{\beta \in X_\Lambda : \mu^q y \notin \beta\}$; clearly, $a \in U$. Since Λ is a ring with identity, $U = \{\beta \in X_\Lambda : (\mu^q y) \cdot \Lambda \not\subset \beta\}$, where $(\mu^q y) \cdot \Lambda$ is the principal ideal generated by $\mu^q y$. Consequently, U is open by the definition of the Zariski topology. It is obvious that for every $\beta \in U$, y will be an annihilator of x.

Proposition 2. *The totality S_T^q of elements of F^q that vanish near $T \subset X_\Lambda$ forms a submodule in F^q;*

$$d^q\big|_{S_T^q} : S_T^q \to S_T^{q+1}.$$

It is clear that it suffices to verify Proposition 2 for the case $S_{\{a\}}^q$, where $\{a\}$ is a one-element subset of X_Λ. Let $x, y \in S_{\{a\}}^q$; $\lambda, \mu \in K$; and let u, v be annihilators of x and y, respectively. Since μ^q is an algebra homomorphism,

$\mu^q(uv) = \mu^q(u)\mu^q(v)$, and hence $\mu^q(uv) \notin a$, since a is a prime ideal, and by property 1 of annihilators,

$$uv \cdot (\lambda x + \mu y) = \lambda v(ux) + \mu u(vy) = 0,$$

therefore $\lambda x + \mu y \in S^q_{\{a\}}$. Let $x \in S^q_{\{a\}}$, and let u be its annihilator. The formula

$$\mu^{q+1} d_i^q = \mu^q, \qquad i = 0, 1, \dots, q+1,$$

is evident; from it it follows that

$$\mu^{q+1} d_i^q(u) = \mu^q(u) \notin a.$$

It is also clear that

$$d_i^q(u) \cdot d_i^q(x) = d_i^q(ux) = 0,$$

whence it follows that

$$u' = \prod_{i=0}^{q+1} d_i^q(u)$$

will be an annihilator of $d^q(x)$. Consequently,

$$d^q x \in S^{q+1}_{\{a\}},$$

as was required.

Corollary. $S_T = \{S_T^q, d^q|_{S_T^q}\}$ is a subcomplex of the complex F.

Denote by $C_T = \{C_T^q\}$ the quotient complex F/S_T.

Definition 2. We set

$$H^q(C_T) = H^q(\Lambda; T); \qquad H^*(\Lambda; T) = \bigoplus_{q=0}^{\infty} H^q(\Lambda; T).$$

It is clear that in the case $T = X_\Lambda$, $H^q(\Lambda; X_\Lambda)$ are covariant functors from the category indicated in the introduction to the category of K-modules. Let $T \subset X_\Lambda$; then $S^q_{X_\Lambda} \subset S^q_T$. We have an exact triple of complexes

$$0 \to S_T/S_{X_\Lambda} \to C_{X_\Lambda} \to C_T \to 0.$$

Put $C_{(X_\Lambda, T)} = S_T/S_{X_\Lambda}$ and

$$H^q(\Lambda; X_\Lambda, T) = H^q(C_{(X_\Lambda, T)}).$$

Then, evidently, the following holds.

Theorem 1. *There is an exact cohomology sequence*

$$\leftarrow H^{q+1}(\Lambda; X_\Lambda, T) \leftarrow H^q(\Lambda; T) \leftarrow H^q(\Lambda; X_\Lambda) \leftarrow H^q(\Lambda; X_\Lambda, T) \leftarrow \cdots.$$

§ 2. Let

$$F = \bigoplus_{q=0}^{\infty} F^q; \qquad C_T = \bigoplus_{q=0}^{\infty} C_T^q.$$

Introduce in F a multiplication operation defined by the following formula for homogeneous elements:

$$x_0 \otimes \cdots \otimes x_p \smile y_0 \otimes \cdots \otimes y_q = x_0 \otimes \cdots \otimes x_p y_0 \otimes \cdots \otimes y_q.$$

The following properties of the operation $\smile$ are obvious. The operation $\smile$ is an associative multiplication in F with identity $1 \in F^0$; if x and y are homogeneous elements of degrees p and q, respectively, then, obviously, the degree of $x \smile y$ is $p+q$, and

$$d(x \smile y) = dx \smile y + (-1)^p x \smile dy.$$

This formula follows easily from the definitions of the operation $\smile$ and of the differential d in F. The operation $\smile$ turns F into an associative K-algebra.

Proposition 3. *In the associative K-algebra F, S_T is a two-sided ideal (obviously, homogeneous).*

It will be sufficient to show this in the case $T = \{a\}$. Let x have degree p and $x \in S^p_{\{a\}}$, and let y be any element of F^q. Let u be an annihilator of x; consider

$$\tilde{x} = x \otimes \underbrace{1 \otimes \cdots \otimes 1}_{q}, \qquad \tilde{y} = \underbrace{1 \otimes \cdots \otimes 1}_{p} \otimes y, \qquad \tilde{u} = u \otimes \underbrace{1 \otimes \cdots \otimes 1}_{q}$$

It is clear that $\mu^{p+q}\tilde{u} = \mu^p u \notin a$, $x \smile y = \tilde{x}\tilde{y}$ ($\tilde{x}$ and $\tilde{y}$ are multiplied according to the multiplication in F^{p+q}); $\tilde{u} \cdot (x \smile y) = (\tilde{u}\tilde{x})\tilde{y} = (ux) \smile y = 0$. Consequently, $S_{\{a\}}$ is a right ideal; it is shown analogously that $S_{\{a\}}$ is a left ideal. Proposition 3 is proved. Thus, the multiplication $\smile$ is defined on every $C_T = F/S_T$ and on $C(X_\Lambda, T) = S_T/S_{X_\Lambda}$.

Theorem 2. *The multiplication $\smile$ defines on $H^*(\Lambda; X_\Lambda, T)$ the structure of an associative skew-commutative K-algebra.*

Consider the automorphism s of the K-module F, defined by the formula $s^q(x_0 \otimes \cdots \otimes x_q) = (-1)^{q(q+1)/2} x_q \otimes \cdots \otimes x_0$. Obviously, for every $T \subset X_\Lambda$ the ideal S_T is invariant with respect to s. It is also not difficult to verify that $ds = sd$. Thus, s is an automorphism of the differential K-module F. Put $\omega_i^q(x_0 \otimes \cdots \otimes x_q) = x_0 \otimes \cdots \otimes x_{i-1} \smile s(x_i \otimes \cdots \otimes x_q)$ for $i = 1, 2, \ldots, q$. It is not difficult to verify that $\omega_i^q : F^q \to F^{q-1}$ is a correctly defined mapping of K-modules. Put

$$h^q = \sum_{i=1}^{q} (-1)^i \omega_i^q.$$

Proposition 4. *The mapping h is a homotopy connecting s and the identity mapping.*

We verify Proposition 4 on the component of degree q, where it takes the form

$$x - s^q(x) = (h^{q+1}d^q)(x) + (d^{q-1}h^q)(x). \tag{1}$$

If $x = x_0 \otimes \cdots \otimes x_q$, then on the right-hand side of the equality being checked one obtains summands of one of the following three types:

a) $x_0 \otimes \cdots \otimes x_{i-1} \otimes 1 \otimes x_i \otimes \cdots \otimes x_{j-1}x_q \otimes \cdots \otimes x_j$;

b) $x_0 \otimes \cdots \otimes x_{j-1}x_q \otimes \cdots \otimes x_i \otimes 1 \otimes x_{i-1} \otimes \cdots \otimes x_j$;

c) $x_0 \otimes \cdots \otimes x_{i-1} \otimes x_q \otimes \cdots \otimes x_i$

with coefficients ±1. Indeed,

$$h^{q+1}d^q + d^{q-1}h^q = \sum_{j=1}^{q+1}\sum_{i=0}^{q+1}(-1)^{i+j}\,\omega_j^{q+1}d_i^q + \sum_{j=1}^{q}\sum_{i=j}^{q}(-1)^{i+j}\,d_i^{q-1}\omega_j^q.$$

Expressions of type a) appear as a result of applying the homomorphisms $\omega_i^{q-1}\omega_j^q$, $\omega_{j+1}^{q+1}d_i^q$ for $0 \leq i \leq j-1 \leq q-1$, with coefficients respectively equal to $(-1)^{i+j+(q-j)(q-j+1)/2}$ and $(-1)^{i+j+1+(q-j)(q-j+1)/2}$, and therefore all summands of type a) cancel pairwise. Expressions of type b) appear from $d_{q-i+j}^{q-1}\omega_j^q$ and $\omega_{j+1}^{q+1}d_i^q$ for $1 \leq j \leq i \leq q$ with coefficients respectively $(-1)^{q-i+j+i+j+(q-j)(q-j+1)/2}$ and $(-1)^{i+j+(q-j+1)(q-j+2)/2}$, or after transformations with coefficients $(-1)^{q-i+(q-j)(q-j+1)/2}$ and $(-1)^{i+q+1+(q-j)(q-j+1)/2}$. Hence it is clear that the terms of type b) also cancel pairwise. Summands of type c) appear after applying homomorphisms of the form $\omega_{i+1}^{q+1}d_i^q$ and $\omega_i^{q+1}d_{q+1}^q$, with coefficients respectively $(-1)^{i+i+1+(q-i)(q-i+1)/2}$ and $(-1)^{i+q+1+(q-i+1)(q-i+2)/2}$, or after transformation with coefficients $(-1)^{2i+1+(q-i)(q-i+1)/2}$ and $(-1)^{2q+2+(q-i)(q-i+1)/2}$, i.e., under the assumption that $1 \leq i \leq q$, all terms of type c) also cancel pairwise. Two terms of type c) remain, obtained as a result of applying the homomorphisms $\omega_1^{q+1}d_0^q$ ($\omega_{i+1}^{q+1}d_i^q$ for $i = 0$) and $\omega_{q+1}^{q+1}d_{q+1}^q$ ($\omega_i^{q+1}d_{q+1}^q$ for $i = q+1$). We have $(\omega_1^{q+1}d_0^q)(x_0 \otimes \cdots \otimes x_q) = (-1)^{q(q+1)/2}x_q \otimes \cdots \otimes x_0$ and $(\omega_{q+1}^{q+1}d_{q+1}^q)(x_0 \otimes \cdots \otimes x_q) = x_0 \otimes \cdots \otimes x_q$. Thus, from the entire right-hand side of expression (1) there remains a sum of the form $x_0 \otimes \cdots \otimes x_q - (-1)^{q(q+1)/2}x_q \otimes \cdots \otimes x_0$, equal to the expression standing on the left-hand side of (1). Hence Proposition 4 is proved.

Let now x and y be cocycles of degrees p and q, respectively; then $x \smile y$ is also a cocycle, which, by Proposition 4, is cohomologous to $s(x \smile y)$. However, we have

$$s(x \smile y) = (-1)^{(p+q)(p+q+1)/2}(y_q \otimes \ldots \otimes y_0 x_p \otimes \ldots \otimes x_0) =$$

$$= (-1)^{(p+q)(p+q+1)/2-p(p+1)/2-q(q+1)/2}(s(y) \smile s(x)).$$

The last expression is cohomologous to $(-1)^{pq} y \smile x$, since, by Proposition 4, the cocycles $s(y)$, $s(x)$ are cohomologous to the cocycles y and x, respectively. This completes the proof of Theorem 2.

§ 3. Let $K = R$ be the field of real numbers and let Λ be the algebra of real continuous functions on a paracompact space X. Let T be the set of ideals in Λ, each of which consists of all functions having zero at some fixed point (in the bicompact case of X we have $T = X_\Lambda$).

Theorem 3. *The algebra $H^*(\Lambda; T)$ is isomorphic to the Alexander–Čech cohomology algebra $\check{H}^*(X; R)$.*

From the definition of C_T and Proposition 1 it follows easily that the elements of C_T are sections over the diagonal ΔX^{p+1} of the space X^{p+1} of germs of continuous functions, representable as sums of a finite number of products of functions of one variable. Let $f(x_0, \dots, x_p)$ be a function representing some section of the indicated type. Then the differential, obviously, acts on it by the formula

$$(df)(x_0, \dots, x_{p+1}) = \sum_{i=0}^{p+1} (-1)^i f(x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_{p+1}).$$

Thus, the definition of cohomology considered here differs from the Alexander–Spanier definition only in that, instead of arbitrary functions, only functions of the indicated type are considered. It is not difficult to verify that the operation of $\smile$-multiplication coincides with the analogous operation in Alexander–Spanier cochains. Therefore there is a homomorphism of algebras

$$i^* : H^*(\Lambda; T) \to \check{H}^*(X; R),$$

induced by the inclusion of sheaves of graded differential germs

$$i : \mathcal{F}'(X; R) \subset \mathcal{F}(X; R).$$

Here $\mathcal{F}(X; R)$ is the graded sheaf of germs of Alexander–Spanier cochains ((2), p. 156), and $\mathcal{F}'(X; R)$ is the sheaf of germs of functions of the type indicated above. The sheaf $\mathcal{F}(X; R)$ is soft ((2), p. 206); the sheaf $\mathcal{F}'(X; R)$ is also soft, as a module over the soft sheaf ${\mathcal{F}'}^0(X; R)$. Therefore ((2), Theorem 3.5.3, Part II) the quotient sheaf

$$\mathcal{F}''(X; R) = \mathcal{F}(X; R)/\mathcal{F}'(X; R)$$

is also soft. Since the inclusion i is compatible with the identity mapping of the constant sheaf R over X, $\mathcal{F}''(X; R)$ may be regarded as a soft resolution of the zero sheaf; consequently, the complex of sections of the sheaf $\mathcal{F}''(X; R)$ is acyclic. It follows that i^* is an isomorphism of groups in each dimension, and since i^* is a homomorphism of algebras, i^* is an isomorphism of algebras.

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Received
22 III 1968

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Note: Figure translations are in progress. See original paper for figures.

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