ON SOME APPLICATIONS OF THE PRINCIPLE OF COMPARISON OF INDICES AND OF SEPARABILITY LAWS OF THE SPACE \(J^{\omega_{\nu}}\)

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Abstract

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Z. I. KOZLOVA

ON SOME APPLICATIONS OF THE PRINCIPLE OF COMPARISON OF INDICES AND OF SEPARABILITY LAWS OF THE SPACE J^{ω_ν}

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We shall consider functions defined at all points of the basic space $\mathfrak{R}$ and taking values in some ordered set $T = (\beta)$. The class of such functions $\Lambda = (\beta(x))$ is called a class of transfinite functions, or a class of indices, if:
1) for whatever homeomorphic transformation $y = \varphi(x) : \mathfrak{R} \to \mathfrak{R}$, from $\beta(x) \in \Lambda$ it follows that $\beta(\varphi(x)) = \beta'(x) \in \Lambda$;
2) from $\beta(x) \in \Lambda$ and $\beta'(x, y) = \beta(x)$ for every $y \in \mathfrak{R}_y$ it follows that $\beta'(x, y) \in \Lambda$.

Let $\beta_1 * \beta_2$ be some order relation between elements of the set T (for example, $\beta_1 = \beta_2$, $\beta_1 \geqslant \beta_2$, $\beta_1 \neq \beta_2$). We shall say that the class of transfinite indices Λ satisfies the θ-principle of comparison of indices with defining relation $*$, if, whatever $\beta_1(x), \beta_2(x) \in \Lambda$ may be, the set $[\beta_1(x) * \beta_2(x)]$ of points x at which the relation $\beta_1(x) * \beta_2(x)$ holds is a set of class θ.

Example 1. Let the space J^{ω_ν} be ordered lexicographically. Add to this space one more point $\{\omega_\nu, 0, 0, \dots, 0, \dots\}$, which we shall regard as following all points of the given space. Let $E \in (B_\alpha) \subset \mathfrak{R} J^{\omega_\nu}_{xy}$. Put $\beta(x, y) = y$, if $(x, y) \in E$; $\beta(x, y) = \{\omega_\nu, 0, 0, \dots, 0, \dots\}$, if $(x, y) \in CE$. Let $\beta_1(x, y)$ be the index of the set $E_1 \in (B_\alpha)$, $\beta_2(x, y)$ the index of the set $E_2 \in (B_\alpha)$, for any $\alpha < \omega_{\nu+1}$; $[\beta_1(x, y) * \beta_2(x, y)] = [\beta_1(x, y) \leqslant \beta_2(x, y)]$. Then the set $V = [\beta_1(x, y) * \beta_2(x, y)] \in (B_\alpha)$, i.e. the family of transfinite functions $\Lambda = (\beta(x, y))$ satisfies the B_α-principle of comparison of indices with defining relation $*$.

Example 2. Let
$\beta(x, y) = (\omega_\nu) T_{\mathfrak{M}} \operatorname{Ind}(x | \{\Delta_i\})$, where $\mathfrak{M} = (M_i) \cup \bigcup (M_{(i_\alpha)\gamma})$ is an $(\omega_\nu) R_M$-family of bases; $\Phi_M \equiv \bigcup_i$; $(\Delta_i)_i$ is a regular family of Baer rectangles of the space $J^{\omega_\nu}_{xy}$; $[\beta_1(x, y) * \beta_2(x, y)] = [\beta_1(x, y) < \beta_2(x, y)]$. Then the set $V = [\beta_1(x, y) * \beta_2(x, y)] \in CT\mathfrak{M}(B)$, where (B) is the class of B-sets of the space $J^{\omega_\nu}_{xy}$, i.e. the class of transfinite functions $\Lambda = (\beta(x, y))$ satisfies the $(\omega_\nu)CA$-principle of comparison of indices with defining relation $*$. Hence it follows that if

$$[\beta_1(x,y) * \beta_2(x,y)] = [\beta_1(x,y) \neq \beta_2(x,y),\ \beta_2(x,y) < \omega_{\nu+1}],$$

i.e.

$$V = [\beta_1(x,y) * \beta_2(x,y)] = ([\beta_1(x,y) < \beta_2(x,y)] \cup [\beta_2(x,y) < \beta_1(x,y)]) \cap [\beta_1(x,y) < \omega_{\nu+1}],$$

and also, if $\beta_2(x,y) = \min\{\beta_1(x,y)+1, \omega_{\nu+1}\}$,

$$[\beta_1(x,y) * \beta_2(x,y)] = [\beta_1(x,y) < \beta_2(x,y)],$$

i.e.

$$V \equiv [\beta_1(x,y) * \beta_2(x,y)] = [\beta_1(x,y) < \beta_2(x,y)],$$

then the class of transfinite functions $\Lambda = (\beta(x,y))$ satisfies the $(\omega_\nu)CA$-principle of comparison of indices with the corresponding defining relation $*$.

Let to a point $x \in \mathfrak{R}$ there be assigned a set $H_x \subset \mathfrak{R}$, and let

$$U = [\beta_1(x) * \beta_2(x') \mid x' \in H_x],$$

i.e.

$$x \in U \equiv (\forall x' \in H_x)[\beta(x) * \beta_2(x')].$$

Let $K(\Lambda)$ be the class of sets representable in the form $[\beta(x) = \omega_{\nu+1}]$. A. A. Lyapunov ([1]) proved the following proposition:

Lemma. Let $\Lambda = (\beta(x))$ be a class of indices, regular ([2,3]) relative to the class $K(\Lambda)$, satisfying the $CK(\Lambda)$- or $BK(\Lambda)$-principle

comparison of indices with defining relation $*$, where the class $K(\Lambda)$ is such that, if $M \in K(\Lambda)$, then $\operatorname{pr}_x M \in K(\Lambda)$, and let to each point x of some set $E \in BK(\Lambda)$ there be assigned a set H_x in such a way that $\bigcup_x x \times H_x \in BK(\Lambda)$. Then, if $\beta_1(x) \in \Lambda,\ \beta_2(x) \in \Lambda$, the set

$$U = [\beta_1(x) * \beta_2(x') \mid x' \in H_x]$$

is a set of class $CK(\Lambda)$.

We note a number of propositions following from this lemma.

1. The union of the Mazurkiewicz sets of all constituents of an $(\omega_\nu)CA$-set of the space $J_{xy}^{\omega_\nu}$ is also an $(\omega_\nu)CA$-set.

2. The Mazurkiewicz set of any B_α-set of the space $J_{xy}^{\omega_\nu}$, for $\alpha < \omega_{\nu+1}$, is a CA_α-set.

3. The set of points of minimal index of every nonempty $(\omega_\nu)CA$-set of the space $J_{xy}^{\omega_\nu}$ is an $(\omega_\nu)CA$-set.

4. The set of points of transfinite uniqueness of an $(\omega_\nu)CA$-set of the space $J_{xy}^{\omega_\nu}$ is an $(\omega_\nu)CA$-set.

5. The set of points of unique index of an $(\omega_\nu)CA$-set of the space $J_{xy}^{\omega_\nu}$ is an $(\omega_\nu)CA$-set.
6. There exists an $(\omega_\nu)CA$-surface $S \subset J_{xyz}^{\omega_\nu}$, uniform for all $(\omega_\nu)CA$-curves.

Let $E \subset \mathfrak{R}_{xy}$ be a set of some class K, all of whose sections parallel to a given direction possess some structural property $\bar{S}$. The question arises whether there exists such a BK-set $\theta \supset E$, all of whose sections parallel to the same direction possess the structural property $\bar{S}$. Problems of this kind have been called problems on the covering of plane sets. They were solved for the class of A-sets of the Baire space J by various authors ([4–9]). Recently A. A. Lyapunov obtained, in the theory of operations on sets, two general theorems on the covering of plane sets. They are formulated for a countable space of indices, but carry over without change to an arbitrary space of indices. Let $P_x = x \times J_y^{\omega_\nu}$.

First theorem on the covering of plane sets. Let H be an $\mathfrak{A}_1$-regular and $\bar{H}$-topological property of bases ([10]), and let $E \subset J_{xy}^{\omega_\nu}$ be an $(\omega_\nu)A$-set. The set of all sets $E \cap P_x$ possessing the property $\bar{H}$, $E_{\bar{H}} = E$. Then there exists a B-set $S \supset E$ such that all sets $S \cap P_x$ possess the property $\bar{H}$.

Second theorem on the covering of plane sets. Let H be an $\mathfrak{A}_1$-regular and $\bar{H}$-topological property of bases; let $E \subset J_{yx}^{\omega_\nu}$ be an $(\omega_\nu)A$-set, and $E_{\bar{H}} \subset E$ the union of all sets $E \cap P_x$ possessing the property $\bar{H}$. Then there exists a set $S \supset E_{\bar{H}}$ of class $(\omega_\nu)CA_{\aleph_0}$ such that all sets $S \cap P_x$ possess the property $\bar{H}$.

We have proved the $\mathfrak{A}_1$-regularity of the properties H_p $(2 \leq p < \omega)$, $H_{\aleph_\sigma}$, $H_{t'}$ for $\tau = \aleph_\nu \leq \aleph_\nu = \tau'$, τ a strongly inaccessible cardinal number ([10]). We also note the $\mathfrak{A}_1$-regularity of the following properties: $H_{\aleph_0}$ (to contain an uncountable number of different chains of the base $\mathfrak{A}_1$ (for $\tau = \aleph_0$—the theorem of C. Mazurkiewicz and W. Sierpiński ([11]), the author ([12])); $H_{\mathrm{clr}\,\alpha}$ (to contain a family of chains of the given base that is not a dispersed set of bounded index $\leq \alpha$ (the author ([8]) for $\tau = \aleph_0$, Z. I. Evdokimova for $\tau > \aleph_0$); $H_C(H_{|C|})$ (to contain a family of chains of the given base that is not τ-compact (with τ-compact closure) (the author ([8]) for $\tau = \aleph_0$, Z. I. Evdokimova for $\tau > \aleph_0$); $H_{\mathrm{abs}\,\alpha}$ (to contain a family of chains of the given base that is not a dispersed family of τ-compact sets of bounded index $\leq \alpha$ (the author ([9]) for $\tau = \aleph_0$, Z. I. Evdokimova for $\tau > \aleph_0$); $H_{\mathrm{red}\,\alpha}$ (to contain a set of chains of the given base that is not a reducible set of index $\leq \alpha$ (Z. I. Evdokimova)).

The formulated theorems on the covering of plane $(\omega_\nu)A$-sets are valid—

valid for the following topological properties $\bar{H}$: $\bar{H}_2$, $\bar{H}_p$ $(2 < p < \omega)$, $\bar{H}$, $\bar{H}_{[c]}$, $\bar{H}_{\mathrm{red}\,\alpha}$.

For the space $J_{xx}^{\omega_\nu}$ the theorems proved by the author ([8,9]) for the space J also hold:

I. Every $(\omega_\nu)A$-set $\mathcal{E} \subset J_{xy}^{\omega_\nu}$ for which all the sets $\mathcal{E} \cap P_x$ are scattered sets of index $\leq \alpha < \omega_{\nu+1}$ can be covered by the same kind of B-set of this space.

II. Every $(\omega_\nu)A$-set $\mathcal{E} \subset J_{xy}^{\omega_\nu}$ for which each of the sets $\mathcal{E} \cap P_x$ is the union of a scattered family of sets with τ-compact closure of index $\leq \alpha < \omega_{\nu+1}$ can be covered by a B-set $H \subset J_{xy}^{\omega_\nu}$, for which all sets $H \cap P_x$ are sets of absolute first class of subclass not exceeding α.

III. Every $(\omega_\nu)A$-set $\mathcal{E} \subset J_{xy}^{\omega_\nu}$ for which all the sets $\mathcal{E} \cap P_x$ are sets of absolute first class of subclass $\leq \alpha < \omega_{\nu+1}$ can be covered by the same kind of B-set of this space.

Let us note that the closure $E^{(y)}$ of an $(\omega_\nu)A$-set $E \subset J_{xy}^{\omega_\nu}$ in the direction $J_{xy}^{\omega_\nu}$ is an $(\omega_\nu)A$-set.

A. A. Lyapunov, in the case of a countable space of indices, also proved general theorems on sections, projection, and representation of B-sets. They also hold for the space $J_{xy}^{\omega_\nu}$.

Section theorem. Let $\bar{H}$ be an $\mathfrak{A}_1$-regular and $\bar{H}$-topological property of bases; $E \subset J_{xy}^{\omega_\nu}$ is a B-set. Then the set $E_H \subset E$, which is the union of all sets $E \cap P_x$ possessing the property $\bar{H}$, is an $(\omega_\nu)CA$-set.

Projection theorem. If $\bar{H}$ is an $\mathfrak{A}_1$-regular and $\bar{H}$-topological property of bases, $E \subset J_{xy}$ is a B-set, $E_{\bar{H}} \subset E$ is the union of all sets $E \cap P_x$ possessing the property $\bar{H}$, then $\operatorname{pr}_x E_{\bar{H}} \subset (\omega_\nu)CA$.

Representation theorem. If $E \subset J_{xy}^{\omega_\nu}$ is such a B-set that the set of all points of the set $E \cap P_x$ possessing the topological property $\bar{H}$, $E_{\bar{H}} = E$, and moreover the property of bases H is $\mathfrak{A}_1$-regular, then $\operatorname{pr}_x E \in (B)$.

As the property $\bar{H}$, one may take any of the topological properties listed above, and also the property $\bar{H}_F$ (to be a closed set) in the first two theorems.

Volgograd Pedagogical Institute
named after A. S. Serafimovich

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Note: Figure translations are in progress. See original paper for figures.