Reports of the Academy of Sciences of the USSR

MATHEMATICS

1969

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR
1969. Volume 186, No. 4

UDC 513.88+517.432:517.43

MATHEMATICS

Dzh. E. ALLAKHVERDIEV

ON NON-SELF-ADJOINT OPERATORS RATIONALLY DEPENDING ON A SPECTRAL PARAMETER

(Presented by Academician M. V. Keldysh on 29 VII 1968)

In the present paper some sufficient conditions are established for the completeness of the system of eigen and associated (e.a.) elements of non-self-adjoint operators rationally depending on the spectral parameter λ in a certain Hilbert space $\mathcal{H}$. In addition, in the case when there are two poles, a theorem is proved on the splitting of the system of e.a. elements into singly complete systems. From these results one obtains some new theorems even for operators depending polynomially on the spectral parameter.

Let the operator $A(\lambda)$ be an analytic function of λ, let a be its pole, and suppose that the estimate

$$\varepsilon_n(A(\lambda)) \le \varepsilon_n + \varepsilon_n^2 |\lambda - a|^{-1} + \dots + \varepsilon_n^{(k)} |\lambda - a|^k$$

holds ($n = N, \dots, \infty$), where

$$\varepsilon_n(A(\lambda)) = \inf \|A(\lambda) - A_n\|,$$

the infimum being taken over all n-dimensional operators A_n in the space $\mathcal{H}$. Put

$$Q = \left\{ \alpha; \sum_{n=1}^{\infty} \varepsilon_n^{\alpha} < \infty \right\}, \qquad \rho = \inf \alpha, \quad \alpha \in Q.$$

Theorem 1. If $[E - A(\lambda)]^{-1}$ exists at least at one point of a sufficiently small neighborhood of the point a and Q is nonempty, then $[E - A(\lambda)]^{-1}$ is a meromorphic operator-function of $|\lambda - a|^{-1}$ (for sufficiently small $|\lambda - a|$) of order not exceeding ρ and of minimal type for order ρ, if $\rho \in Q$.

Consider the operator

$$C(\mu) = \mu R_1(\mu) A_1 + \mu^2 R_2(\mu) A_2 + \dots + \mu_n^{nR}(\mu) A_n.$$

Put

$$\varepsilon_n = \max_k [\varepsilon_n(A_k)]^{1/k}.$$

Theorem 2. If Q is nonempty and the resolvent of $C(\mu)$ exists for some μ_0 ($|\mu_0| > M$), then it is a meromorphic function in the domain $|\mu| > M$ of order not exceeding ρ and of minimal type for order ρ, if $\rho \in Q$.

Let $\Gamma_a(\lambda)$ be the principal part of the Laurent expansion of the operator $A(\lambda)$ in a neighborhood of the pole a. From this theorem it is easy to obtain that the order and type of the resolvent of the operator $A(\lambda)$ in a neighborhood of its pole a do not exceed the order and type of the resolvent of $\Gamma_a A(\lambda)$ in a neighborhood of the point a.

In what follows we shall say:

The operator $A(\lambda)$ satisfies **condition A** in a neighborhood of its pole a, if $\Gamma_a A(\lambda)$ has the form

$$\Gamma_a A(\lambda) = \sum_{k=1}^{n_a-1} (\lambda - a)^{-k} A_{a,k} A_a^k + (\lambda - a)^{-n_a} T_a^{n_a},$$

where T_a is a complete operator of finite order r_a, the $A_{a,k}$ are arbitrary completely continuous operators, and there exists a finite system of rays R_a, issuing from the origin of coordinates, such that the angle between neighboring rays of R_a is less than π/r_a, and on each ray of the system, for sufficiently large λ, the condition

$$\|(E - \lambda^{n_a} T_a^{n_a})^{-1} T_a^i\| \leq \frac{\alpha}{|\lambda|^i} \qquad (i = 0, \dots, n_a - 1)$$

is satisfied.

$A(\lambda)$ satisfies **condition B** if

$$\Gamma_a A(\lambda) = \sum_{k=1}^{n_a-1} (\lambda - a)^{-k} H_a^{\alpha k} B_{a,k} H_a^{(1-\alpha)k} + (\lambda - a)^{-n_a} H_a^{n_a},$$

where H_a is a complete normal operator of finite order, $B_{a,k}$ are arbitrary completely continuous operators, and the eigenvalues (e.v.) of the operator λH_a lie inside a finite number of strips.

$A(\lambda)$ satisfies **condition C** if

$$\Gamma_a A(\lambda) = \sum_{k=1}^{n_a-1} (\lambda - a)^{-k} C_{a,k} H_a^k + \sum_{k=0}^{n_a} (\lambda - a)^{-k} z_{a,k} H_a^k,$$

where H_a is a complete self-adjoint operator of finite order; $z_{a,k}$ are arbitrary complex numbers, $z_{a,k} \neq 1$, $z_{a,n_a} \neq 0$, and $C_{a,k}$ are arbitrary completely continuous operators.

$A(\lambda)$ satisfies **condition D** if $\Gamma_a A(\lambda)$ is a finite-dimensional operator.

Conditions A-D are naturally extended also to the pole $\lambda = \infty$.

Let the operator $A(\lambda)$ have as singular points only a finite number of poles in the λ-plane and $\lambda = \infty$ be a pole of $A(\lambda)$ satisfying one of the conditions A, B, C. Then the following theorem is valid.

Theorem 3. *Let the operator $A(\lambda)$, in a neighborhood of each pole, satisfy at least one of the conditions A, B, C, D, and let the poles satisfying condition D be regular for $[E - A(\lambda)]^{-1}$. Then: 1) the system of root vectors is N-fold complete in $\mathcal{H}$ (N is the sum of the multiplicities of all poles satisfying at least one of the conditions A, B, C); 2) the poles satisfying one of the conditions A, B, C, and only they, are limit points of the eigenvalues of the operator $A(\lambda)$; 3) in neighborhoods of poles satisfying conditions B or C, for any ε there exists only a finite number of eigenvalues which satisfy none of the inequalities $a_i - \varepsilon \leq \arg(\lambda - a)^{-1} \leq a_i + \varepsilon$, where: in case B, a_i is the argument of the rays along which the eigenvalues of the operator $\lambda^{n_a} H_a^{n_a}$ are located; in case C, a_i is the argument of the rays passing through the origin and the roots of the equation*

$$\sum_{k=1}^{n_a} z_{a,k} \mu^k = \pm 1.$$

If some of the poles satisfying condition D are not regular for $[E - A(\lambda)]^{-1}$, then the orthogonal complement of the derived system of root vectors corresponding to all eigenvalues λ_k, $\lambda_k \neq a_{is}$, where a_{is} $(s = 1, 2, \dots, p)$ are the poles in neighborhoods of which the operator $A(\lambda)$ satisfies condition D, is a finite-dimensional subspace in $\mathcal{H}^{(N)}$*.

In the case when $\lambda = \infty$ is not a pole of the operator $A(\lambda)$, the theorem remains valid if, in addition, the existence of $[E - A_{00}]^{-1}$ or $[E - \Pi_{a_i} A(a_i)]^{-1}$ is required for at least one i.

Let us note that for a derived chain $\psi_{i,h} = \{\psi_{i,h}^{r,j}\}$ (see ([3], [4])) of any operator of the form

$$A(\lambda) = \sum_{i=1}^{n} \sum_{k=1}^{m_i} \frac{A_{i,k}}{(\lambda - a_i)^k} + \sum_{k=0}^{m_0} \lambda^i A_{0k},$$

where $A_{i,k}$ $(i = 0, \dots, n;\ k = 0, 1, \dots, m_i)$ are arbitrary completely continuous operators, su-

* In general, from the elements which constitute the kernel of the finite-dimensional operators that are the coefficients of the principal parts of the resolvent in a neighborhood of its poles (see ([1]), pp. 11-12, the form of the resolvent), an N-fold complete system is constructed. However, those of the indicated elements which correspond to the poles of the resolvent coinciding

with finite-dimensional poles of the operator $A(\lambda)$ may be neither proper nor associated elements for $A(\lambda)$ corresponding to the eigenvalues $\lambda = a_{is}$. In general, when a is an irregular point for the operators, the notion of an eigenvalue $\lambda = a$ and of root vectors corresponding to the eigenvalue $\lambda = a$ must be defined additionally.

there exists a system $\Phi_{l,t} = \{\Phi^{r,j}_{l,t}\}$, for which the equalities $[\psi_{i,h}\Phi_{l,t}] = \delta_{i,l}\delta_{h,t}$ hold, where $[X, Y]$ is the scalar product in the space $\mathcal{H}^{(N)}$ ($\mathcal{H}^{(N)}$ is the topological product of $N = \sum_{j=0}^{n} m_j$ copies of the Hilbert space $\mathcal{H}$). Hence it follows that if in the space $\mathcal{H}^{(N)}$ there exists an expansion with respect to the derived system of eigen- and associated elements of the operator $A(\lambda)$, then the coefficients of the expansion are determined uniquely.

Now consider an operator of the form

$$A(\lambda) = \sum_{i=1}^{m} \lambda^{-i} A_i + \sum_{k=0}^{n-1} \lambda^k H^k B_k + \lambda^n H^n, \tag{1}$$

where H is a complete positive self-adjoint operator, A_i $(i = 1, \dots, m)$ are bounded operators, B_k $(k = 0, \dots, n-1)$ are completely continuous operators. All operators act in the Hilbert space $\mathcal{H}$.

Theorem 4. *Suppose that one of the following conditions is satisfied:*

1) $\lim\limits_{n\to\infty} n|\mu_n|^{-\rho} = 0;\quad A^{-\rho}_{iH},\ B^{-\rho}_{kH}$ $(i = 1, \dots, m;\ k = 0, \dots, n-1)$ *are bounded operators;*

2) $\lim\limits_{n\to\infty} n|\mu_n|^{-\rho} < \infty,\quad A^{-\rho}_{iH}$ $(1, \dots, m)$ *are bounded, while* $B^{-\rho}_{kH}$ $(k = 0, \dots, n-1)$ *are completely continuous operators,*

where μ_n *are the eigenvalues of the operator* μH, *and* ρ *is some positive number. Then the subspaces* $\mathcal{H}_k$, *formed by a certain subsystem of eigen- and associated elements of the operator* (1), *corresponding to eigenvalues lying in each of the regions*

$$G_k(r, \varphi, \psi) = \{\lambda :\ |\lambda| \ge r,\ a_k - \varphi \le \arg\lambda \le a_k + \psi,\ \varphi < \varepsilon,\ \psi < \varepsilon\},$$

are quadratically close to the subspaces $\mathcal{H}'_k$, *formed by the eigen-elements of the operator* $\lambda^n H^n$, *corresponding to eigenvalues lying in* $G_k(r, \varphi, \psi)$ *(i.e.* $|\lambda_i| \ge r,\ \arg\lambda_i = 2\pi k n^{-1}$*).*

If to the indicated system of subspaces, for sufficiently large r, one adds the subspace composed of the eigen-elements of the operator λ^{nH^n} corresponding to eigenvalues λ_i with $|\lambda_i| < r,\ \arg\lambda_i = 2\pi i n^{-1}$, then a Bari basis of subspaces* in the space $\mathcal{H}$ is obtained.

Lemma 1. *If ω is a linearly independent system of finite-dimensional subspaces in $\mathcal{H}_i$ which is quadratically close to the orthogonal system of subspaces $\mathcal{H}'_i$, then there exists a completely continuous operator A such that*

$$\mathcal{H}'_i = (E - A)\mathcal{H}_i.$$

In the case when the dimensions of all subspaces $\mathcal{H}_i$ (or $\mathcal{H}'_i$) are bounded by one number, the operator A becomes a Hilbert-Schmidt operator. From $\mathcal{H}'_i = (E - A)\mathcal{H}_i$ it follows that $\{\mathcal{H}'_i\}$ is a Riesz basis of subspaces of a very special kind, which, apparently, possesses many good properties.

In the proof of Theorem 4 the following lemma was used, which also has independent significance.

Lemma 2. *Let P be the projection operator of some subspace $\mathcal{H}_1 = P\mathcal{H}$, and let a linear operator K satisfy the condition $\|K - P\| < \varepsilon$, $\varepsilon < 1$. Then there exists a subspace $\mathcal{H}_2$ such that $\mathcal{H}_2 \subset K\mathcal{H}$ and the operator Q, orthogonally projecting onto $\mathcal{H}_2$, satisfies the condition*

$$\|P - Q\| < \varepsilon(1 - \varepsilon)^{-1}.$$

An analogue of this lemma is also valid in a Banach space, if in this space for each subspace G a projection operator P is defined satisfying certain additional conditions.

Let us note that for one very important class of operators polynomially dependent on λ, a theorem on estimating the order of growth of the resolvent was ob-

* For the definition of a Bari basis of subspaces and of quadratic closeness of systems of subspaces, see ([11]), p. 289, and ([14]).

was obtained by M. V. Keldysh in connection with the work [(1)] (see [(4,5)]); and, in the case of an operator depending linearly on λ, analogous estimates were obtained by Carleman for operators of Hilbert-Schmidt type (see also [(5,7)]). Theorems analogous to Theorems 1 and 2 are also valid in a Banach space. Using estimates of $\varepsilon_n(A(\lambda))$ and V. I. Macaev' s results on estimating the resolvent of operators depending linearly on λ, one can obtain analogous, more precise estimates of the growth of the resolvent for operators depending nonlinearly on λ. This remark is also valid for a Banach space. Theorem 3 generalizes the author' s results [(3,4)], and in the case of a single pole at an infinitely distant point it yields new theorems on the completeness of the system of root vectors for operators depending polynomially on λ.

I express my gratitude to Academician M. V. Keldysh, Professor M. A. Naimark, and Professor V. B. Lidskii for their attention to this work and for discussion of the results.

Institute of Cybernetics
Academy of Sciences of the Azerbaijan SSR

Received
25 VII 1968

REFERENCES

[1] M. V. Keldysh, DAN, 77, No. 1, 11 (1951).
[2] M. V. Keldysh, V. B. Lidskii, Tr. IV Vsesoyuzn. matem. s' ezda, 1, 1963, p. 101.
[3] Dzh. E. Allakhverdiev, DAN, 160, No. 6 (1965).
[4] Dzh. E. Allakhverdiev, DAN, 166, No. 1 (1966).
[5] V. B. Lidskii, Tr. Mosk. matem. obshch., 2 (1962).
[6] T. Carleman, Math. Zs., 9, 196 (1921).
[7] E. Hill, J. D. Tamarkin, Acta Math., 57, 1 (1931).
[8] V. I. Macaev, DAN, 154, No. 5 (1964).
[9] V. I. Macaev, DAN, 155, No. 2 (1964).
[10] R. M. Dzhabarzade, Uch. zap. AGU, No. 3 (1964).
[11] V. A. Vizitei, A. S. Markus, Matem. sborn., 66 (108), 2.
[12] M. Schimbrot, Arch. Rat. Mech. and Anal., 15, No. 5 (1964).
[13] H. Weyl, Proc. Nat. Acad. Sci. U.S.A., 35, 408 (1949).
[14] I. Ts. Gokhberg, M. G. Krein, Introduction to the Theory of Linear Nonselfadjoint Operators, Moscow, 1965.
[15] Dzh. E. Allakhverdiev, DAN, 115, No. 2 (1957).

Note: Figure translations are in progress. See original paper for figures.