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Abstract

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MATHEMATICS

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ON THE THEORY OF SINGULAR INTEGRAL EQUATIONS WITH MEASURABLE COEFFICIENTS

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Let a simple closed oriented Lyapunov contour Γ divide the complex plane into an exterior domain D^- and an interior domain D^+, and let the point $z = 0$ belong to D^+. By Λ we denote the set of functions piecewise continuous on Γ, and by $\mathcal{L}$ the closure of this set in the sense of uniform convergence. Following ([1]), we shall call functions in $\mathcal{L}$ lineatchatye.

In the present note we investigate the problem of perturbations of a singular operator with bounded measurable coefficients and establish a general theorem on the solvability of singular integral equations in the space L_p with coefficients from $\mathcal{L}$.

Consider the singular integral equation in the space $L_p(\Gamma)$

$$d(t)(S\varphi)(t) + c(t)\varphi(t) = g(t), \tag{1}$$

where $c(t)$ and $d(t)$ are measurable and essentially bounded functions on Γ, and

$$(S\varphi)(t) = \frac{1}{\pi i}\int_\Gamma \frac{\varphi(\tau)}{\tau - t}\,d\tau \quad (t \in \Gamma).$$

With the aid of the projectors $P = (I + S)/2$ and $Q = (I - S)/2$, the integral equation (1) is rewritten in the following form:

$$aP + bQ = g,$$

where $a(t) = c(t) + d(t)$, $b(t) = c(t) - d(t)$.

Theorem 1. *Let $a(t)$ be a bounded measurable function and let the operator $A = aP + Q$ be a Φ-operator in the space $L_p(\Gamma)$. Then there exists a number r $(1 < r < \infty)$ such that, for any function $c(t)$ of the class $A[r]$, the operator $B = acP + Q$ is a Φ-operator, and*

$$\dim\ker B = \max(0, \operatorname{ind} A - \operatorname{ind} c), \qquad \dim\operatorname{coker} B = \max(0, \operatorname{ind} c - \operatorname{ind} A).$$

Proof. For the Φ-operator A there exists a number $\eta > 0$ ([3]) such that any operator A_0 for which $\|A_0 - A\| < \eta$ is a Φ-operator and $\operatorname{ind} A_0 = \operatorname{ind} A$. Choose a number $r(> 1)$ from the condition

$$\max[r, r'] > \pi \sup |a| \, \|P\|/\eta,$$

where $1/r + 1/r' = 1$.

Let now $c(t)$ be any function of the class $A[r]$; then ([2])

$$c(t) = c_1(t)c_2(t),$$

where $c_1(t)$ is a Hölder function and $\operatorname{ind} c_1 = \operatorname{ind} c = \chi$, while $c_2(t)$ is a function of the class $A[r]$ and

$$|\arg c_2| < \pi / \max[r, r'] - \varepsilon.$$

Since for the function

$$m(t) = c_2(t)/|c_2(t)| - 1$$

we have

$$|m(t)| < \sup_{t \in \Gamma} |\arg c_2(t)| < \eta / \sup |a| \, \|P\|,$$

it follows that

$$K = a(1 + m)P + Q$$

is a Φ-operator and $\operatorname{ind} K = \operatorname{ind} A$.

Let $\chi \geq 0$. Using the factorization of the functions $c_1 = f_+ t^\chi f_-$ and $|c_2| = g_+ g_-$, where the functions $f_+ f_-$, g_+, g_- and their inverses are bounded on the contour Γ (see ([2])), the operator B can be represented in the form

$$B = X(a(1 + m)P + Q)Y(t^\chi P + Q),$$

where

$$X = g_- f_- I$$

and

$$Y = g_+ f_+ P + g_-^{-1} f_-^{-1} Q$$

are invertible

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* For the definition of the class $A[r]$ and the index of a function $c(t) \in A[r]$, see ([2]).

operators. Taking into account that the operator $t^\chi P + Q$ is a Φ-operator and $\operatorname{ind}(t^\chi P + Q) = -\chi$, we obtain that B is a Φ-operator, $\operatorname{ind} B = \operatorname{ind} A - \chi = \operatorname{ind} A - \operatorname{ind} c$, and by virtue of ([4,5]) it is left invertible.

For $\chi < 0$ the equality

$$B(t^{-\chi} P + Q) = X(a(1 + m)P + Q)Y$$

holds, whence it follows that the operator B is right invertible and $\operatorname{ind} B = \operatorname{ind} A - \chi$. The theorem is proved.

Corollary. If $A = aP + Q$ is a Φ-operator, then there exists a number $l(a,p)$ (>0) such that, for every measurable function $c(t)$ satisfying the conditions $0 < N \le |c(t)| \le M < \infty$ and

$$\sup_{t\in\Gamma} |\arg c(t)| < l(a,p),$$

the operator $B = acP + Q$ will be a Φ-operator and $\operatorname{ind} B = \operatorname{ind} A$.

For the case of a piecewise continuous function $a(t)$, one can indicate the dependence of the quantity $l(a,p)$ on the function $a(t)$ and the space L_p. First let us note some properties of functions from Λ and $\mathcal{L}$.

Following ([6]), to each function $a(t) \in \mathcal{L}$ we associate, in the natural way, the curve with orientation

$$V_p(a)(\mu,t) = a(t-0)M(p,\mu) + a(t+0)(1 - M(p,\mu))$$

$$(0 \le \mu \le 1,\ t \in \Gamma),$$

where

$$M(p,\mu) = \exp\{(1-\mu)\theta\} \sin\mu\theta/\sin\theta, \quad \theta = \pi(1-2/p),$$

if $p \ne 2$, and $V_2(a) = a(t-0)\mu + a(t+0)(1-\mu)$. A function $a(t)$ from $\mathcal{L}$ is called p-nonsingular if

$$\inf_{t,\mu} |V_p(a)| > 0.$$

Let $a(t)$ be a p-nonsingular function from $\mathcal{L}$, t_0 an arbitrary point of continuity, and $t_1, t_2, \dots$ its discontinuity points; then the ratio $a(t_k-0)/a(t_k+0)$ can be represented in the form

$$|a(t_k-0)/a(t_k+0)| \exp\{ih_k(a)\} \quad (k = 0,1,2,\dots),$$

where

$$-2\pi/q < h_k(a) < 2\pi/p, \qquad 1/p + 1/q = 1.$$

To each such function one can assign two numbers $h_+(a)$ and $h_-(a)$, defining them by the equalities

$$h_+(a) = \max_k h_k(a), \qquad h_-(a) = \min_k h_k(a).$$

Obviously,

$$-2\pi/q < h_-(a) \le h_k(a) \le h_+(a) < 2\pi/p. \tag{2}$$

For p-nonsingular functions from $\mathcal{L}$ the index is introduced in the following way. Let a sequence of functions $a_n(t)$ from Λ converge uniformly to $a(t)$. Then, starting from some n, the functions $a_n(t)$ are p-nonsingular and their p-index is equal to one and the same number; this number is taken to be the p-index of the function $a(t)$ from $\mathcal{L}$.

Theorem 2. For a p-nonsingular function $a(t)$ from Λ, as the number $l(a,p)$ one may take

$$\min[2\pi - ph_+(a),\ 2\pi + qh_-(a)]/2\max[p,q] \qquad (1/p + 1/q = 1).$$

Moreover, if $\chi(=\operatorname{ind}_p a) = 0$, the operator $A = acP + Q$ is invertible in the space $L_p(\Gamma)$; if $\chi > 0$, the operator A is left invertible and $\dim\operatorname{coker} A = \chi$; if $\chi < 0$, the operator A is right invertible and $\dim\ker A = |\chi|$.

Remark 1. The number $l(a,p)$ indicated in the theorem is, in a certain sense, sharp. Indeed, take on the unit circle Γ_0 the piecewise continuous functions

$$c = a = \exp\{i\theta/4\} \qquad (-\pi < \theta \le \pi);$$

then

$$\max_{\in\Gamma_0} |\arg c| = l(a,p),$$

and the operator $caP + Q$ in the space $L_2(\Gamma_0)$ is not invertible from either side (see ([6])).

Proof of Theorem 2. From the discontinuity points t_k of the function $a(t)$ and the numbers $h_k(a)$ we construct the function

$$\psi(t) = \prod_{k=1}^{m} t^{\gamma_k}, \qquad \text{where } \gamma_k = \frac{h_k(a)}{2\pi} + \frac{1}{2\pi i}\ln\left|\frac{a(t_k-0)}{a(t_k+0)}\right|,$$

and the function t^{γ_k} has a discontinuity only at the point t_k $(k = 1,2,\dots,m)$. Then the function $b(t) = a(t)\psi^{-1}(t)$ is continuous on the contour Γ, and $\operatorname{ind} b = \operatorname{ind}_p a$. Using the condition of the theorem, choose a number $\varepsilon > 0$ so that

$$|\arg c| < (\min[2\pi - ph_+(Q),\ 2\pi + qh_-(Q)] - \varepsilon)/2\max[p,q].$$

Let $r(t)$ be such–

a rational function such that $b(t) = r(t)(1+m(t))$, $\operatorname{ind} r(t) = \operatorname{ind} b(t)$, $|\arg(1+m)| < \varepsilon/8\max[p,q]$. The functions r, ψ, and $c_1 = c(1+m)$ admit the factorization ([2],[7])

$$r = r_+ t^{\chi} r_-, \qquad \psi = \psi_+\psi_-, \qquad c(1+m) = c_+c_-.$$

The factorization factors for the functions ψ and c_1 can be represented in the form

$$\psi_\pm = F_\pm \rho_1^{\pm 1}, \quad \text{where } \sup_t |F_\pm^{\pm 1}(t)| < \infty, \quad \rho_1(t) = \prod_{k=1}^{m} |t - t_k|^{\operatorname{Re}\gamma_k},$$

$$c_\pm = u_\pm \rho_2^{\pm 1}, \quad \text{where } \sup_t |u_\pm^{\pm 1}(t)| < \infty, \quad \rho_2(t) = \exp\left\{ \frac{1}{2\pi} \int_\Gamma \frac{\arg c_1(\tau)}{\tau - t}\, d\tau \right\}.$$

Consider the operator $Z = (\rho_1\rho_2)^{-1} S \rho_1 \rho_2$; its boundedness in the space $L_p(\Gamma)$ is proved by applying Stein' s theorem ([8]). Indeed, take $\gamma = 2\pi/(\min[2\pi - ph_+(a),\ 2\pi + qh_-(a)] - \varepsilon/2)(> 1)$; then $|\gamma \arg c_1(t)| < \pi/\max[p,q] - \varepsilon_1$ $(\varepsilon_1 > 0)$, and by I. B. Simonenko' s theorem on weights ([2])

$$\|(Sf)\rho_2^{-\gamma}\|_{L_p} \le M_1 \|f\rho_2^{-\gamma}\|_{L_p}. \tag{3}$$

Moreover, since the function $a(t)$ is p-nonsingular, for the chosen γ the relations

$$-1/q < \operatorname{Re}\gamma_k \cdot \gamma/(\gamma - 1) < 1/p \quad (k = 1, 2, \dots, m)$$

follow from (2).

Then from B. V. Khvedelidze's theorem on the boundedness of singular operators in L_p spaces with a weight (([7]), p. 24) it follows that

$$\|(Sf)\rho_1^{-\gamma/(\gamma-1)}\|_{L_p} \le M_2 \|f\rho_1^{-\gamma/(\gamma-1)}\|_{L_p}. \tag{4}$$

Since for $t = 1/\gamma$

$$\rho_2^{-\gamma t} \rho_1^{-(1-t)\gamma/(\gamma-1)} = (\rho_2\rho_1)^{-1},$$

from (3) and (4), by virtue of Stein' s interpolation theory, the boundedness of the operator Z follows.

Let $\operatorname{ind}_p a = \chi = 0$; then the operator A can be represented in the form

$$A = acP + Q = r_-\psi_-c_-(r_+\psi_+c_+P + r_-^{-1}\psi_-^{-1}c_-^{-1}Q).$$

Consider the operator

$$B = (r_+^{-1}\psi_+^{-1}c_+^{-1}P + r_-\psi_-c_-Q)r_-^{-1}\psi_-^{-1}c_-^{-1}.$$

Its boundedness follows easily from the boundedness of the operator Z. It is not difficult to verify that the operator B is inverse to A. Thus, for $\chi = 0$ the operator $acP + Q$ is invertible in $L_p(\Gamma)$.

Let $\chi > 0$; then $\operatorname{ind}(at^{-\chi}) = 0$, and from what was proved above it follows that the operator $t^{-\chi}acP + Q$ is invertible in $L_p(\Gamma)$. Since the operator $t^\chi P + Q$ is left invertible, it follows from the equality

$$(act^{-\chi}P + Q)(t^\chi P + Q) = acP + Q$$

that the operator A is left invertible and $\dim \operatorname{coker} A = \chi$.

If $\chi < 0$, then

$$act^{-\chi}P + Q = (acP + Q)(t^{-\chi}P + Q),$$

and, consequently, the operator A is right invertible and $\dim \ker A = |\chi|$.

Remark 2. The proof given above extends without difficulty to the case of a p-nonsingular function $a(t)$ from $\mathcal{L}$.

Theorem 3. Let $a(t), b(t) \in \mathcal{L}$. In order that the operator

$$\hat{A} = aP + bQ$$

be a $\Phi_+(\Phi_-)$-operator in the space $L_p(\Gamma)$, it is necessary and sufficient that: 1) $\operatorname{ess\,inf}_{t\in\Gamma} |b(t)| > 0$; 2) the function $a(t)/b(t)$ be p-nonsingular.

If these two conditions are fulfilled and $\chi = \operatorname{ind}_p(a/b)$, then for $\chi = 0$ the operator $\hat{A}$ is invertible in the space $L_p(\Gamma)$; for $\chi > 0$ the operator $\hat{A}$ is left invertible and $\chi = \dim \operatorname{coker} \hat{A}$; for $\chi < 0$ the operator $\hat{A}$ is right invertible and $|\chi| = \dim \ker \hat{A}$.

Proof. The sufficiency of the conditions of the theorem follows from Remark 2. Necessity is established as follows. Let $\hat{A} = aP + bQ$ be a $\Phi_+(\Phi_-)$-operator and

$$\operatorname*{ess\,inf}_{t\in\Gamma} |b(t)| = 0.$$

Since the set of $\Phi_+(\Phi_-)$-operators is open, one can choose functions $\tilde{a}$ and $\tilde{b}$ from Λ such that

$$\operatorname*{ess\,inf}_{t\in\Gamma} |\tilde{b}(t)| = 0$$

and $\tilde{a}P + \tilde{b}Q$ is a $\Phi_+(\Phi_-)$-operator, which leads to a contradiction with (6). Thus

$$\operatorname*{ess\,inf}_{t\in\Gamma} |b(t)| > 0.$$

Now the problem reduces to the study of the operator

$$A_0 = (a/b)P + Q.$$

Let A_0 be a $\Phi_+(\Phi_-)$-operator and let a/b not be a p-nonsingular function. Then there exists a function $g(t) \in \Lambda$ such that

$$\inf_{t\in\Gamma,\ 0\le\mu\le 1} |V_p(g)(\mu, t)| = 0$$

and the operator $gP + Q$ is a $\Phi_+(\Phi_-)$-operator. This leads to a contradiction. The theorem is proved.

Let us note that the sufficiency of the conditions of Theorem 3 for the case when

$$\sum |h_k(a)| < \infty$$

is established in ([9]).

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