ON SOME PROBLEMS OF NONLINEAR STOCHASTIC PROGRAMMING

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Abstract

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Problems of planning and control under conditions of incomplete information can be written in terms of stochastic programming (see, for example, ([1])). Published works on stochastic programming concern mainly linear problems. In ([2]) methods are studied for the numerical solution of convex stochastic conditional extremum problems. In the present paper approaches are considered to the formulation and solution of a fairly broad class of nonlinear (including nonconvex) stochastic problems. A number of known models of stochastic programming fit into the proposed scheme.

1°. Let (Ω, F, p) be a probability space on the compact set Ω, and let X, Y be separable B-spaces. To each $\omega \in \Omega$ there are assigned nonempty sets $G_0(\omega), \ldots, G_n(\omega) \subset X$ and a linear operator $A(\omega) : X \to Y$ defined on the whole space X.

It is assumed that: a) all the sets $\{x, \omega : x \in G_i(\omega)\}$, $i = 0, 1, \ldots, n$, are Borel; b) $G_0(\omega) \neq X$ with probability 1; c) $A(\omega)$ is a measurable operator-function, i.e., for any measurable $x(\omega)$ the function $y(\omega) = A(\omega)x(\omega)$ is measurable.

Problem I. It is required to minimize

$$P\{x(\omega) \in G_0(\omega)\} \tag{1}$$

under the conditions

$$P\{x(\omega) \in G_i(\omega)\} \geq \alpha_i, \qquad i = 1, \ldots, n; \tag{2}$$

$$M\{A(\omega)x(\omega)\} \leq b, \qquad b \in Y. \tag{3}$$

In what follows the case $n = 2$ is considered. The same methods are applicable also in the general case. The additional difficulties arising here are of a purely combinatorial nature.

2°. Consider the auxiliary Problem II, obtained from Problem I by discarding condition (3).

Problem II. It is required to minimize

$$P\{x(\omega) \in G_0(\omega)\}$$

under the conditions

$$P\{x(\omega) \in G_0(\omega)\} \geq \alpha_i, \qquad i = 1, 2.$$

The events corresponding to the following nine situations are pairwise incompatible and the probability of at least one of them is equal to 1.

(1) $(G_1 \cap G_2) \setminus G_0 \neq \emptyset$;
(2) $\emptyset \neq G_1 \cap G_2 \subset G_0,\ G_1 \setminus G_0 \neq \emptyset,\ G_2 \setminus G_0 \neq \emptyset$;
(3) $\emptyset \neq G_1 \cap G_2 \subset G_0,\ G_1 \setminus G_0 \neq \emptyset,\ G_2 \setminus G_0 = \emptyset$;
(4) $\emptyset \neq G_1 \cap G_2 \subset G_0,\ G_1 \setminus G_0 = \emptyset,\ G_2 \setminus G_0 \neq \emptyset$;
(5) $\emptyset \neq G_1 \cap G_2 \subset G_0,\ G_1 \setminus G_0 = \emptyset,\ G_2 \setminus G_0 = \emptyset$;
(6) $G_1 \cap G_2 = \emptyset,\ G_1 \setminus G_0 \neq \emptyset,\ G_2 \setminus G_0 \neq \emptyset$;
(7) $G_1 \cap G_2 = \emptyset,\ G_1 \setminus G_0 \neq \emptyset,\ G_2 \setminus G_0 = \emptyset$;
(8) $G_1 \cap G_2 = \emptyset,\ G_1 \setminus G_0 = \emptyset,\ G_2 \setminus G_0 \neq \emptyset$;
(9) $G_1 \cap G_2 = \emptyset,\ G_1 \setminus G_0 = \emptyset,\ G_2 \setminus G_0 = \emptyset$.

Denote by p_i the probability of event (i).

Obviously, for the consistency of problem II it is necessary and sufficient that

$$0 \leq \alpha_i \leq 1 \quad (i = 1, 2), \qquad \alpha_1 + \alpha_2 + p_6 + p_7 + p_8 + p_9 \leq 2. \tag{4}$$

In what follows condition (4) is assumed to be satisfied.
Introduce the functions

$$\varphi_i(x, \omega) = \begin{cases} 1, & x \in G_i(\omega), \\ 0, & x \notin G_i(\omega). \end{cases}$$

Then

$$P\{x(\omega) \in G_i(\omega)\} = \int_\Omega \varphi_i(\omega, x(\omega))\, dp_\omega.$$

Problem II is transformed into an extremal problem with an integral functional and integral constraints, and the methods developed in (3) are applicable to its solution.

Denote by $S_2(\alpha_1, \alpha_2)$ the required lower bound of the functional (1) under the conditions (2).

Theorem 1. $S_2(\alpha_1, \alpha_2)$ is equal to the greatest of the following 7 numbers

$$\begin{aligned}
&(\mathrm{I}) && \alpha_1 + \alpha_2 - 1 - (p_1 - p_9), \\
&(\mathrm{II}) && \alpha_2 - p_1 - p_2 - p_4 - p_6 - p_8, \\
&(\mathrm{III}) && 2\alpha_1 + \alpha_2 - 2 - (p_1 - p_8 - p_9), \\
&(\mathrm{IV}) && \alpha_1 - p_1 - p_2 - p_3 - p_6 - p_7, \\
&(\mathrm{V}) && \alpha_1 + 2\alpha_2 - 2 - (p_1 - p_7 - p_9), \\
&(\mathrm{VI}) && \frac{1}{2}[\alpha_1 + \alpha_2 - 1 - (p_1 - p_5 - p_9)], \\
&(\mathrm{VII}) && 0.
\end{aligned} \tag{5}$$

We shall now clarify the nature of the solution of the problem. Consider 7 equations

$$\text{(I)}\quad \varphi_1+\varphi_2-\varphi_0=\begin{cases}2 & \text{in situation } (1),\\ 1 & \text{in} \quad (2)\div(8),\\ 0 & \text{in} \quad (9);\end{cases}$$

$$\text{(II)}\quad \varphi_2-\varphi_0=\begin{cases}1 & \text{in situations } (1),(2),(4),(6),(8),\\ 0 & \text{in} \quad (3),(5),(7),(9);\end{cases}$$

$$\text{(III)}\quad 2\varphi_1+\varphi_2-\varphi_0=\begin{cases}3 & \text{in situation } (1),\\ 2 & \text{in} \quad (2)-(7),\\ 1 & \text{in} \quad (8),(9);\end{cases}$$

$$\text{(IV)}\quad \varphi_1-\varphi_0=\begin{cases}1 & \text{in situations } (1)\div(3),(6),(7),\\ 0 & \text{in} \quad (4),(5),(8),(9);\end{cases} \tag{6}$$

$$\text{(V)}\quad \varphi_1+2\varphi_2-\varphi_0=\begin{cases}3 & \text{in situation } (1),\\ 2 & \text{in} \quad (2)-(6),(8),\\ 1 & \text{in} \quad (7),(9);\end{cases}$$

$$\text{(VI)}\quad \frac{\varphi_1+\varphi_2}{2}-\varphi_0=\begin{cases}1 & \text{in situation } (1),\\ \frac{1}{2} & \text{in} \quad (2)-(4),(6)-(8),\\ 0 & \text{in} \quad (5),(9);\end{cases}$$

$$\text{(VII)}\quad \varphi_0=0.$$

Theorem 2. Suppose $S_2(\alpha_1,\alpha_2)$ is equal to the i-th number of system (5). Then there exists a solution $x(\omega)$ of problem II satisfying conditions (2) and the i-th equation of system (6).

3°. Let us return to problem I. We shall call a sequence $x_n(\omega)$ nearly admissible if

$$\lim_{n\to\infty} P\{x_n(\omega)\in G_i(\omega)\}\geq \alpha_i, \qquad M(A(\omega)x_n(\omega))=y_n'+y_n'',$$

where $y_n'\leq b,\ y_n''\to 0$.

If the set of almost admissible sequences is nonempty, then problem I is called weakly consistent. With each almost admissible sequence $\{x_n(\omega)\}$ we associate the functional

$$\varliminf_{n\to\infty} P\{x_n(\omega)\in G_0(\omega)\}$$

and denote by $S(a_1, a_2)$ its greatest lower bound over all almost admissible sequences, and by $S_1(a_1, a_2)$ the greatest lower bound of (1) in problem I.

Let L be the set of elements of the space Y that can be represented in the form

$$y = M\{A(\omega)x(\omega)\} = \int_\Omega A(\omega)x(\omega)\, dp_\omega.$$

Obviously, L is a linear manifold.

Theorem 3. Problem I is weakly consistent if and only if problem II is consistent and there exists an element $y \in \overline{L}$ (the closure of L) satisfying the inequality $y \le b$.

If, moreover, Y is finite-dimensional, then weak consistency of problem I is equivalent to its consistency.

Theorem 4. If problem I is weakly consistent, then

$$S(a_1, a_2) = S_2(a_1, a_2).$$

If, moreover, Y is finite-dimensional, then

$$S_1(a_1, a_2) = S_2(a_1, a_2).$$

Let $F \subset \Omega$ be an arbitrary measurable set, and let L_F be the collection of elements $y \in Y$ represented in the form

$$y = \int_F A(\omega)x(\omega)\, dp_\omega.$$

For any $\omega \in \Omega$, denote by $L_\omega = \bigcap L_F$, where the intersection is taken over all neighborhoods F of the point ω. Finally, let $\sum_{\omega\in\Omega} L_\omega$ be the collection of all possible finite sums $y_1 + \cdots + y_n,\ y_i \in L_{\omega_i}$.

Theorem 5. Let problem I be weakly consistent. Then for any $\varepsilon > 0$ there exist a function $x(\omega)$ and a vector $y \in \sum L_\omega,\ y = y_1 + \cdots + y_n,\ y_i \in L_{\omega_i}$, such that:

a) $P\{x(\omega) \in G_0(\omega)\} \le S_1(a_1, a_2) + \varepsilon$;

b) $P\{x(\omega) \in G_i(\omega)\} \ge a_i - \varepsilon, \quad i = 1, 2$;

c) $M\{A(\omega)x(\omega)\} + y = y' + y''$, where $y' \le b,\ \|y''\| < \varepsilon$;

d) $x(\omega)$, with probability $1 - \varepsilon$, satisfies Theorem 2.

Corollary. Let $Y = R^n$ and let problem I be consistent. Then, if $x_0(\omega)$ is a solution of problem II, there exist points $\omega_1, \dots, \omega_r$ ($r \le n$) and vectors $y_1, \dots, y_r$, $y_i \in L_{\omega_i}$, such that

$$M\{A(\omega)x_0(\omega)\} + \sum y_i \le b.$$

From the last corollary and the definition of L_ω, the structure of almost admissible minimizing sequences for problem I follows at once. These sequences are formed by functions $x_n(\omega)$ representable as the sum

$$x_n(\omega) = x_0(\omega) + \sum_i x_{in}(\omega),$$

where $x_{in}(\omega)$ is different from zero only in some neighborhood of the point ω_i, having measure not exceeding $1/n$, and

$$\int_\Omega A(\omega)x_{in}(\omega)\,dp_\omega = y_i, \qquad i = 1, 2, \dots, r \le n.$$

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References

1. E. G. Gol' shtein, D. B. Yudin, *New Directions in Linear Programming*, Moscow, 1966.

2. D. B. Yudin, *Economics and Mathematical Methods*, No. 6 (1968).

3. A. D. Ioffe, V. M. Tikhomirov, *Doklady Akademii Nauk*, 184, No. 4 (1968).

Note: Figure translations are in progress. See original paper for figures.