

# ON THE CLASSICAL SOLUTION OF A MULTIDIMENSIONAL MULTIFRONT STEFAN PROBLEM

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## Abstract

## Full Text

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## MATHEMATICS

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# ON THE CLASSICAL SOLUTION OF A MULTIDIMENSIONAL MULTIFRONT STEFAN PROBLEM

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In the present note we prove the existence, uniqueness, and stability with respect to perturbations of the initial data of a regular classical solution of a multidimensional multifront Stefan problem (Problem A) with internal and external phase fronts. The exposition is given for the case when the number of spatial independent variables is  $N = 2$ , but the method and the results are easily extended to the case  $N > 2$ .

**1°.** **Problem A.** Find the functions  $u^i = u^i(x_1, x_2, t)$ ,  $x_1 = x_1^i(s, t)$ ,  $x_2 = x_2^i(s, t)$ ,  $0 \leq s \leq s_0$ ,  $0 \leq t \leq T$ ,  $i = 1, 2, \dots, n$ , from the conditions

$$u_t^i = a_i^2 (u_{x_1 x_1}^i + u_{x_2 x_2}^i) + F^i(x_1, x_2, t) \quad \text{for } (x_1, x_2, t) \in D_T^i, \quad i = 1, \dots, n, \quad (1)$$

where  $\bar{D}_T^i$  is a closed simply connected domain of values of  $(x_1, x_2, t)$ , whose boundary consists of four simply connected plane pieces  $D_{0x_1x_2}^i$ ,  $D_{Tx_1x_2}^i$ ,  $D_{tx_10}^i$ ,  $D_{tx_1l_{22}}^i$ , lying respectively in the planes  $t \equiv 0$ ,  $t \equiv T$ ,  $x_2 \equiv l_{12} = 0$ ,  $x_2 \equiv l_{22}$ , and two curved simply connected pieces

$$S_T^j = \{x_1 = x_1^j(s, t), x_2 = x_2^j(s, t), 0 \leq s \leq s_0, 0 \leq t \leq T\},$$

$j = i, i + 1$ ;  $i = 1, 2, \dots, n$ , with  $S_T^k$ ,  $k = 1, \dots, n$ , being the unknown surfaces (phase fronts), and  $S_T^{n+1}$  a given surface;  $D_T^i = \bar{D}_T^i \setminus \Gamma_T^i$ ,

$$\Gamma_T^i = D_{0x_1x_2}^i \cup D_{tx_10}^i \cup D_{tx_1l_{22}}^i \cup S_T^i \cup S_T^{i+1};$$

$$u^i(x_1, x_2, 0) = \varphi_i(x_1, x_2), \quad (x_1, x_2) \in D_{0x_1x_2}^i, \quad i = 1, 2, \dots, n; \quad (2)$$

$$u^k \Big|_{S_T^i} = u^k(x_1, x_2, t) \Big|_{x_1=x_1^i(s,t), x_2=x_2^i(s,t)} = \omega_k^i(s, t) = \omega_k^i(x_1^i(s, t), x_2^i(s, t), t);$$

$$i = k, k + 1; \quad k = 1, \dots, n; \quad \omega_{k-1}^k \equiv \omega_k^k, \quad k = 2, \dots, n - 1. \quad (3')$$

$$u^k|_{D_{tx_1l_{j_2}}^k} = u^k(x_1, x_2, t)|_{x_2=l_{j_2}} = f_k^j(x_1, t), \quad j = 1, 2;$$

$$f_k^j(x, t) \text{ is defined in } D_{tx_1l_{j_2}}^k; \quad j = 1, 2; \quad k = 1, \dots, n; \quad (3'')$$

$$-\frac{\partial x_i^j(s, t)}{\partial t} = \bar{\lambda}^{jj-1}(s, t) \frac{\partial u^{j-1}(x_1^j(s, t), x_2^j(s, t), t)}{\partial x_i} - \bar{\lambda}^{jj}(s, t) \frac{\partial u^j(x_1^j(s, t), x_2^j(s, t), t)}{\partial x_i}, \quad (4)$$

$$i = 1, 2; \quad j = 1, 2, \dots, n, \quad \text{where } u_{x_1}^0 \equiv 1, \quad u_{x_2}^0 \equiv 1;$$

$$\bar{\lambda}^{jk} = \lambda^j(s, t, x_1^k(s, t), x_2^k(s, t)), \quad k = j - 1, j; \quad j = 1, 2, \dots, n; \quad (5)$$

$$x_i^j(s, 0) = \psi_i^j(s), \quad 0 \leq s \leq s_0, \quad i = 1, 2; \quad j = 1, 2, \dots, n + 1.$$

Under the assumption that the initial curves

$$\Gamma_j = \{x_1 = \psi_1^j(s), x_2 = \psi_2^j(s)\}, \quad j = 1, 2, \dots, n + 1,$$

have no common points, we shall seek a solution of Problem A ((1) – (5)) on such a time interval  $0 \leq t \leq T^*$  on which  $S_{T^*}^i$ ,  $i = 1, \dots, n + 1$ , have no common points and the condition

$$[x_{1s}^i(s, t)]^2 + [x_{2s}^i(s, t)]^2 \neq 0 \quad (6)$$

is satisfied.

**Definition 1.** The surface  $S_T^i \equiv \{x_1 = x_1^i(s, t), x_2 = x_2^i(s, t), 0 \leq s \leq s_0, 0 \leq t \leq T\}$  is said to satisfy requirements B if:

- 1)  $S_T^i$  is a simple unclosed smooth surface; in particular,  $x_{1t}^i(s, t)$ ,  $x_{2t}^i(s, t)$ ,  $x_{1s}^i(s, t)$ ,  $x_{2s}^i(s, t)$  are continuous and

$$[x_{1s}^i(s, t)]^2 + [x_{2s}^i(s, t)]^2 \neq 0$$

for  $0 \leq s \leq s_0$ ,  $0 \leq t \leq T$ , and all interior points of  $S_T^i$  are interior points of the cylinder

$$C \equiv \{0 \leq x_2 \leq l_{22}, 0 \leq t \leq T, -\infty < x_1 < +\infty\};$$

- 2) the boundary of the surface  $S_T^i$  is a simple continuous, closed piecewise-smooth curve consisting of four smooth pieces lying respectively on the faces  $x_2 = 0$ ,  $x_2 = l_{22}$ ,  $t = 0$ ,  $t = T$  of the cylinder  $C$ .

**Definition 2.** A regular classical solution of problem A is a system of functions

$$u^i = u^i(x_1, x_2, t), \quad x_1 = x_1^i(s, t), \quad x_2 = x_2^i(s, t), \quad 0 \leq s \leq s_0, \quad 0 \leq t \leq T, \quad i = 1, 2, \dots, n,$$

satisfying the conditions: 1) the surfaces  $S_T^i$  satisfy requirements B and have no common points with one another or with  $S_T^{n+1}$ ; 2)  $u^i = u^i(x_1, x_2, t)$  are continuous in  $\overline{D_T^i}$ ,  $i = 1, 2, \dots, n$ ;  $u_{x_1}^i(x_1, x_2, t)$ ,  $u_{x_2}^i(x_1, x_2, t)$ ,  $u_{x_1 x_1}^i(x_1, x_2, t)$ ,  $u_{x_1 x_2}^i(x_1, x_2, t)$ ,  $u_{x_2 x_2}^i(x_1, x_2, t)$ ,  $u_t^i(x_1, x_2, t)$  are continuous in  $D_T^i$ ,  $i = 1, 2, \dots, n$ , and take uniformly continuous limiting values at the interior points of the surfaces  $S_T^j$ ,  $j = i, i + 1$ ; 3) all relations (1)–(5) of problem A and condition (6) are satisfied.

**Remark.** If  $(u^i, x_1^i, x_2^i, i = 1, \dots, n)$  is a regular classical solution of problem A, then it is also a classical solution of problem A in the following sense: on each phase front  $S_T^i$ , represented by the equation  $\Phi^i(x_1, x_2, t) = 0$  (which is possible in a neighborhood of each point  $\in S_T^i$ , since by the condition

$$[x_{1s}^i(s, t)]^2 + [x_{2s}^i(s, t)]^2 \neq 0$$

for  $0 \leq s \leq s_0$ ,  $0 \leq t \leq T$ ,  $i = 1, 2, \dots, n$ ), the relation

$$\Phi_t^i + (\lambda^{ii-1} \text{grad } u^{i-1} - \lambda^{ii} \text{grad } u^i, \text{grad } \Phi^i) = 0, \quad i = 1, 2, \dots, n.$$

$2^0$ . We reduce problem A to a system of nonlinear integral equations of the second kind. To this end consider the functions

$$G_{1k}(x_1, t, \xi_1, \tau) = \frac{1}{2a_k \sqrt{\pi(t-\tau)}} \exp \left[ -\frac{(x_1 - \xi_1)^2}{4a_k^2(t-\tau)} \right], \quad k = 1, 2, \dots, n;$$

$$G_{2k}(x_2, t, \xi_2, \tau) = \frac{1}{2a_k \sqrt{\pi(t-\tau)}} \sum_{n=-\infty}^{+\infty} \left\{ \exp \left[ -\frac{(x_2 - \xi_2 + 2nl_{22})^2}{4a_k^2(t-\tau)} \right] - \exp \left[ -\frac{(x_2 + \xi_2 - 2nl_{22})^2}{4a_k^2(t-\tau)} \right] \right\}, \quad k = 1, \dots,$$

$$\Pi_k(x_1, x_2, t, \xi_1, \xi_2, \tau) = G_{1k}(x_1, t, \xi_1, \tau) G_{2k}(x_2, t, \xi_2, \tau), \quad k = 1, \dots, n; \quad (7)$$

$$u_m^{kl}(s, t) = u_{x_m}^k(x_1^l(s, t), x_2^l(s, t), t), \quad g_{mr}^{kl}(s, t) = u_{x_m x_r}^k(x_1^l(s, t), x_2^l(s, t), t), \quad (8)$$

$$m, r = 1, 2; \quad l = k, k + 1; \quad k = 1, 2, \dots, n.$$

Integrating Green's identity for  $\Pi_k$  and  $u^k$  over  $D_t^k$  from (1), (2), and (3), we obtain, for the regular classical solution  $u^k$  of problem A, the representation

$$u^k(x_1, x_2, t) = \iint_{D_{0x_1 x_2}^k} \varphi_k(\xi_1, \xi_2) \Pi_k d\xi_1 d\xi_2 - a_k^2 \sum_{i=1}^2 \iint_{D_{tx_1^i t_2^i}^k} f_k^i(\xi_1, \tau) \Pi_{k\xi_2} d\xi_1 d\tau +$$

$$\begin{aligned}
 & + \sum_{i=k}^{k+1} \iint_{S_t^i} \bar{\omega}_k^i(s, \tau) \Pi_k d\xi_1 d\xi_2 - a_k^2 \sum_{j=1}^2 \sum_{i=k}^{k+1} \iint_{S_t^i} \bar{\omega}_k^i(s, \tau) \Pi_{k\xi_j} d\xi_{p(j)} d\tau + \\
 & + a_k^2 \sum_{j=1}^2 \sum_{i=k}^{k+1} \iint_{S_t^i} v_j^{ki}(s, \tau) \Pi_k d\xi_{p(j)} d\tau + \iiint_{D_t^k} F_k(\xi_1, \xi_2, \tau) \Pi_k d\xi_1 d\xi_2 d\tau, \\
 & p(j) = 2 - 2^{-1}[1 + (-1)^j], \quad k = 1, \dots, n. \tag{9}
 \end{aligned}$$

Next, consider the system (10)–(12) of nonlinear Volterra integral equations of the second kind for the functions  $v_m^{kl}(s, t)$ ,  $x_i^k(s, t)$ ,  $x_{is}^k(s, t)$ ,  $i = 1, 2$ ;  $l = k, k + 1$ ;  $k = 1, 2, \dots, n$

$$\begin{aligned}
 v_m^{kl} = & \iint_{D_{0x_1x_2}^k} \varphi_k \Pi_{kx_m} d\xi_1 d\xi_2 - a_k^2 \sum_{i=1}^2 \iint_{D_{tx_1t_i^2}^k} f_k^i \Pi_{k\xi_2x_m} d\xi_1 d\tau + \\
 & + \sum_{i=k}^{k+1} \iint_{S_t^i} \bar{\omega}_k^i \Pi_{kx_m} d\xi_1 d\xi_2 - a_k^2 \sum_{j=1}^2 \sum_{i=k}^{k+1} \iint_{S_t^i} \bar{\omega}_k^i \Pi_{k\xi_jx_m} d\xi_{p(j)} d\tau + \\
 & + \sum_{j=1}^2 \sum_{i=k}^{k+1} \iint_{S_t^i} v_j^{ki} \Pi_{kx_m} d\xi_{p(j)} d\tau + \iiint_{D_t^k} F_k \Pi_{kx_m} d\xi_1 d\xi_2 d\tau; \tag{10}
 \end{aligned}$$

$$p(j) = 2 - 2^{-1}[1 + (-1)^j], \quad m = 1, 2; \quad l = k, k + 1; \quad k = 1, \dots, n;$$

$$\begin{aligned}
 x_i^j(s, t) = & \psi_i^j(s) + \int_0^t \left[ \bar{\lambda}^{jj-1}(s, \tau) v_i^{jj-1}(s, \tau) - \bar{\lambda}^{jj}(s, \tau) v_i^{jj}(s, \tau) \right] d\tau, \\
 & j = 1, \dots, n; \quad i = 1, 2; \tag{11}
 \end{aligned}$$

$$\begin{aligned}
 x_{is}^j(s, t) = & \psi_{is}^j(s) + \int_0^t \left\{ \bar{\lambda}_s^{jj-1}(s, \tau) v_i^{jj-1}(s, \tau) - \bar{\lambda}_s^{jj}(s, \tau) v_i^{jj}(s, \tau) + \right. \\
 & \left. + \sum_{k=j-1}^j (-1)^{j-k+1} \bar{\lambda}^{jk}(s, \tau) \left[ g_{i1}^{kj}(s, \tau) x_{1s}^j(s, \tau) + g_{i2}^{kj}(s, \tau) x_{2s}^j(s, \tau) \right] \right\} d\tau, \\
 & i = 1, 2; \quad j = 1, \dots, n; \tag{12}
 \end{aligned}$$

$$\lambda_s^{jk}(s, t) = \frac{d}{ds} \lambda^j(s, t, x_1^k(s, t), x_2^k(s, t)); \quad (13)$$

$$\begin{aligned} g_{mr}^{kl}(s, t) = & \iint_{D_{0x_1x_2}^k} \varphi_k \Pi_{k\xi_2 x_{mx}r} d\xi_1 d\xi_2 - a_k^2 \sum_{i=1}^2 \iint_{D_{tx_1t_i^2}^k} f_k^i \Pi_{k\xi_1 x_{mx}r} d\xi_1 d\tau + \\ & + \sum_{i=k}^{k+1} \iint_{S_i^i} \bar{\omega}_k^i \Pi_{kx_{mx}r} d\xi_1 d\xi_2 - a_k \sum_{j=1}^2 \sum_{i=k}^{k+1} \iint_{S_i^i} \bar{\omega}_k^i \Pi_{k\xi_j x_{mx}r} d\xi_{p(j)} d\tau - \\ & - a_k^2 \sum_{j=1}^2 \sum_{i=k}^{k+1} \iint_{S_i^i} v_j^{ki} \Pi_{kx_{mx}r} d\xi_{p(j)} d\tau + \iiint_{D_t^k} F_k \Pi_{kx_{mx}r} d\xi_1 d\xi_2 d\tau, \quad (14) \end{aligned}$$

$$m, r = 1, 2; \quad l = k, k + 1; \quad p(j) = 2 - 2^{-1}[1 + (-1)^j]; \quad k = 1, 2, \dots, n;$$

$$g_{mr}^{kl}(s, t) = g_{rm}^{kl}(s, t).$$

**Definition 3.** Problem A (1)–(5) is equivalent to system B (10)–(12) if every regular classical solution of problem A (1)–(5) generates, by virtue of (8) and (9), a continuous solution of system B (10)–(12), and conversely every continuous solution of system B (10)–(12), satisfying inequality (6), generates, by virtue of (8) and (9), a regular classical solution of problem A (1)–(5).

**Theorem 1 (equivalence).** Suppose that for the initial data of problem A the following conditions are satisfied:

- 1)  $F^k(x_1, x_2, t)$ ,  $F_{x_i}^k(x_1, x_2, t)$ ,  $F_t^k(x_1, x_2, t)$  are continuous with respect to  $x_1, x_2, t$ ,  $k = 1, 2, \dots, n$ ;
- 2)  $\varphi_k(x_1, x_2)$ ,  $\varphi_{kx_i}(x_1, x_2)$ ,  $\varphi_{kx_ix_j}(x_1, x_2)$ ,  $k = 1, 2, \dots, n$ ;  $i, j = 1, 2$ , are continuous in  $x_1, x_2$ ;
- 3)  $f_k^i(x_1, t)$ ,  $f_{kx_1}^i(x_1, t)$ ,  $f_{kt}^i(x_1, t)$ ,  $f_{kx_1t}^i(x_1, t)$ ,  $i = 1, 2$ ;  $k = 1, 2, \dots, n$ , are continuous in  $x_1, t$ ;
- 4)  $\bar{\lambda}^{kh}(s, t)$ ,  $\bar{\lambda}^{kh-1}(s, t)$ ,  $\bar{\omega}^{ki}(s, t)$  are continuously differentiable with respect to all arguments;
- 5)  $\psi_i^j(s)$ ,  $\psi_{is}^j(s)$ ,  $j = 1, \dots, n$ ;  $i = 1, 2$ , are continuous in  $s$ ;  $[\psi_{1s}^j(s)]^2 + [\psi_{2s}^j(s)]^2 = 0$ ,  $j = 1, \dots, n + 1$ ;  $0 \leq s \leq s_0$ ;
- 6)  $x_i^{n+1}(s, t)$ ,  $x_{is}^{n+1}(s, t)$ ,  $x_{it}^{n+1}(s, t)$ ,  $i = 1, 2$ , are continuous;  $[x_{1s}^{n+1}(s, t)]^2 + [x_{2s}^{n+1}(s, t)]^2 \neq 0$ ,  $0 \leq s \leq s_0$ ,  $0 \leq t \leq T$ ,  $i = 1, 2$ .

7) The compatibility conditions for the initial and boundary data are satisfied.

Then problem A (1)–(6) is equivalent to the system of integral equations B (10)–(12), by virtue of relations (13), (14).

**Theorem 2.** Under the conditions of Theorem 1 there exists an interval of time  $0 \leq t \leq T^*$ ,  $0 < T^* \leq T$ , on which there exists a unique regular solution of system B (10)–(12), and hence there exists a unique regular classical solution of problem A (1)–(5).

The proof of Theorem 2 is obtained by the method of contractions.

**Definition 4.** Suppose that, along with problem A (1)–(5), there is given problem  $A^*$  (1\*)–(5\*), differing from problem A (1)–(5) only in that all the initial data  $\varphi_k(x_1, x_2)$ ,  $f_k^i(x_1, t)$ ,  $F_k(x_1, x_2, t)$ ,  $a_k$ ,  $\psi_i^k(s)$ ,  $\bar{\lambda}^{kk-1}(x_1, x_2, s, t)$ ,  $\bar{\lambda}^{kk}(x_1, x_2, s, t)$ ,  $\bar{\omega}^{ki}(s, t)$  are replaced, respectively, by  $\varphi_k^*(x_1, x_2)$ ,  $f_k^{i*}(x_1, t)$ ,  $F_k^*(x_1, x_2, t)$ ,  $a_k^*$ ,  $\psi_i^{k*}(s)$ ,  $\bar{\lambda}^{kk-1*}(x_1, x_2, s, t)$ ,  $\bar{\lambda}^{kk*}(x_1, x_2, s, t)$ ,  $\bar{\omega}^{ki*}(s, t)$ . Put

$$\eta = \max \left\{ \max_k |a_k - a_k^*|, \max_{s,i,k} |\psi_1^k(s) - \psi_i^{k*}(s)|, \max_{s,i,k} |\psi_{is}^k(s) - \psi_{is}^{k*}(s)|, \right. \\ \max_{x_1, x_2, s, t, k} |\bar{\lambda}^{kk-1} - \bar{\lambda}^{kk-1*}|, \max_{x_1, x_2, s, t, k} |\bar{\lambda}^{kk} - \bar{\lambda}^{kk*}|, \\ \max_{x_1, x_2, k} |\varphi_k - \varphi_k^*|, \max_{x_1, x_2, i, k} |\varphi_{kx_i} - \varphi_{kx_i}^*|, \max_{x_1, x_2, i, j, k} |\varphi_{kx_ix_j} - \varphi_{kx_ix_j}^*|, \\ \max_{x_1, i, k, t} |f_k^i(x_1, t) - f_k^{i*}(x_1, t)|, \max_{t, x_1, i, k} |f_{kx_1}^i(x_1, t) - f_{kx_1}^{i*}(x_1, t)|, \\ \max_{t, x_1, i, k} |f_{kt}^i(x_1, t) - f_{kt}^{i*}(x_1, t)|, \max_{t, x_1, i, k} |f_{kx_1t}^i(x_1, t) - f_{kx_1t}^{i*}(x_1, t)|, \\ \left. \max_{x_1, x_2, t, k} |F_k - F_k^*|, \max_{x_1, x_2, t, i, k} |F_{kx_i} - F_{kx_i}^*|, \max_{x_1, x_2, t} |F_{kt} - F_{kt}^*| \right\}, \\ 0 \leq t \leq T^*, \quad 0 < T^* \leq T,$$

$$\varepsilon = \max \left[ \max_{i, k, s, 0 \leq t \leq T^*} |x_i^k(s, t) - x_i^{k*}(s, t)|, \max_{s, i, k, 0 \leq t \leq T^*} |x_{is}^k(s, t) - x_{is}^{k*}(s, t)|, \right. \\ \left. \max_{x_1, x_2, k, 0 \leq t \leq T^*} |u^k(x_1, x_2, t) - u^{k*}(x_1, x_2, t)| \right].$$

The regular classical solution  $u^k(x_1, x_2, t)$ ,  $x_1^k(s, t)$ ,  $x_2^k(s, t)$ ,  $k = 1, 2, \dots, n$ ,  $0 \leq s \leq s_0$ ,  $0 \leq t \leq T^*$ ,  $0 < T^* \leq T$ , of problem A (1)–(5) is called **stable with respect to perturbations of the initial data** if  $\varepsilon \rightarrow 0$  as  $\eta \rightarrow 0$ .

**Theorem 3.** If the initial data of problems A and  $A^*$  satisfy the conditions of Theorem 1, then the regular classical solution of problem A is stable with respect to perturbations of the initial data.

**Remark.** All the results extend also to a quasilinear parabolic equation with principal part

$$u_t^k = a_k^2 (u_{x_1 x_1}^k + u_{x_2 x_2}^k)$$

and with a free term depending on  $x_1, x_2, t, u^k, u_{x_1}^k, u_{x_2}^k$ , to quasilinear boundary conditions, and also to the case where, along with unclosed phase fronts, there are closed phase fronts.

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