AXIOMATIC FIELD THEORY IN TERMS OF OPERATOR JACOBI MATRICES

PHYSICS

1969

SovietRxiv

Abstract

Full Text

UDC 517.948.35+539.126.01:539.126

PHYSICS

Yu. M. BEREZANSKII, V. D. KOSHMANENKO

AXIOMATIC FIELD THEORY IN TERMS OF OPERATOR JACOBI MATRICES

(Presented by Academician S. L. Sobolev on 9 IV 1969)

In this note it is shown that the axiomatic theory of a Hermitian scalar field can be completely reformulated in terms of operator Jacobi matrices; moreover, only one restriction, apparently of little essential significance, is imposed on the domains of definition of the field operators. In the particular case of free and generalized free fields, the entries of these Jacobi matrices are creation and annihilation operators; a situation analogous in form also arises in the general case. The passage from Wightman functionals to operator Jacobi matrices is analogous to the well-known passage from the power moment problem to Jacobi matrices.

1°. Let H be a Hilbert space. We shall consider a Hermitian scalar field $A(\varphi)$ ([1,2]). This means that a linear mapping of the Schwartz space of test functions $S(R_4) \ni \varphi \mapsto A(\varphi)$ is given into the space of linear operators $A(\varphi)$ acting in H (here R_4 is the Minkowski space of points $x = (x^0, x^1, x^2, x^3)$). The field operators $A(\varphi)$ are defined on a common dense domain of definition D in H, with $A(\varphi)D \subseteq D$, $A^*(\varphi) \supseteq A(\bar{\varphi})$, and for any $\Phi, \Psi \in D$, $l_{\Phi,\Psi}(\cdot) = (A(\cdot)\Phi, \Psi) \in S'(R_4)$. Thus, $A(\varphi)$ is a generalized operator function over $S(R_4)$; formally one may write

$$A(\varphi) = \int_{R_4} A(x)\varphi(x)\,dx.$$

The following axioms are required to hold. **1) Existence of the vacuum.** There exists a vector $\Omega \in H$ (the vacuum) such that $\mathfrak{A}\Omega = \{A\Omega;\ A \in \mathfrak{A}\}$ is dense in H, where $\mathfrak{A}$ is the algebra generated by $\{A(\varphi);\ \varphi \in S(R_4)\}$ and 1. Below, as D we take the "minimal" domain of definition $\mathfrak{A}\Omega$. **2) Relativistic invariance.** In H there is given a fixed continuous unitary representation $(a, \Lambda) \mapsto U(a, \Lambda)$ of the inhomogeneous proper orthochronous Lorentz group $\mathcal{P}_+^\uparrow$, carrying D into D, and such that

$$U(a,\Lambda)A(\varphi)U^{-1}(a,\Lambda) = A(\varphi_{(a,\Lambda)}) \quad (\varphi_{(a,\Lambda)}(x) = \varphi(\Lambda^{-1}(x-a)), \quad \varphi \in S(R_4), \quad (a,\Lambda) \in \mathcal{P}_+^\uparrow).$$

3) Spectrality. The commutative group of unitary operators $U(a, 1)$ $(a \in R_4)$, by Stone' s theorem, admits the representation

$$U(a,1) = \int_{R_4} \exp i(p,a)\, dE(p),$$

where $dE(p)$ is a certain four-dimensional resolution of the identity ($(\cdot,\cdot)$ is the indefinite scalar product in R_4). It is required that $E(\Delta)$ be concentrated in

$$V_+^\varepsilon = \{p \in R_4 :\ (p,p) \geq \varepsilon > 0,\ p^0 > 0\}$$

and at the point $p = 0$, with $E(\{0\})$ equal to the projector onto the vacuum Ω.
4) Local commutativity. If the supports $\operatorname{supp}\varphi_1$ and $\operatorname{supp}\varphi_2$ are compact and spacelike separated (i.e. $(x_1 - x_2, x_1 - x_2) < 0$ for any $x_1 \in \operatorname{supp}\varphi_1$, $x_2 \in \operatorname{supp}\varphi_2$), then the commutator

$$[A(\varphi_1), A(\varphi_2)] = 0.$$

We shall additionally require that the following condition be satisfied. Denote

$$\mathfrak{A}_n = \text{l.o. } \{A(\varphi_n) \cdots A(\varphi_1);\ \varphi_1, \dots, \varphi_n \in S(R_4)\} \quad (n = 1, 2, \dots),$$

$$\mathfrak{A}_0 = \{\lambda 1,\ \lambda \in C\}$$

(l.o. = linear span). Obviously, $A(\varphi)$ acts from $\mathfrak{A}_n\Omega$ into $\mathfrak{A}_{n+1}\Omega$ $(n = 0, 1, \dots)$; it is required that this action be continuous (this, of course, does not mean continuity of $A(\varphi)$ in the whole of H). Now

$A(\varphi)$ can be extended by continuity to an operator acting continuously from $\mathfrak{H}_n$ to $\mathfrak{H}_{n+1}$, where $\mathfrak{H}_m$ is equal to the closure of $\mathfrak{A}_m\Omega$; in what follows we assume that such an extension has been made.

2°. For a given field $A(\varphi)$ let us construct a decomposition of $\overline{H}$ into an orthogonal sum of subspaces. Put $H_0 = \mathfrak{H}_0$, $H_1 = \text{c.l.s. } \{\mathfrak{H}_0, \mathfrak{H}_1\} \ominus \mathfrak{H}_0, \dots \dots, H_n = \text{c.l.s. }$ $\{\mathfrak{H}_0, \dots, \mathfrak{H}_n\} \ominus$ c.l.s. $\{\mathfrak{H}_0, \dots, \mathfrak{H}_{n-1}\}, \dots$ (c.l.s. = closed linear span). Obviously,

$$\text{c.l.s. } \{\mathfrak{H}_0, \dots, \mathfrak{H}_n\} = \bigoplus \sum_{j=0}^{n} H_j, \qquad H = \bigoplus \sum_{j=0}^{\infty} H_j.$$

The domain of definition D of the operators $A(\varphi)$ coincides with the set of finite sums

$$\bigoplus \sum_{j=0}^{n} H_j;$$

the restriction of $A(\varphi)$ to $\overline{H}_j$ acts in

$$\overline{H}_{\max(j-1,0)} \oplus H_j \oplus H_{j+1} :$$

obviously,

$$A(\varphi) H_j \subseteq \bigoplus \sum_{k=0}^{j+1} H_k,$$

and let now $\Phi \in H_j$ and

$$\Psi \in \sum_{k=0}^{j-2} H_k.$$

Then

$$(A(\varphi)\Phi, \Psi) = (\Phi, A^*(\varphi)\Psi) = (\Phi, A(\overline{\varphi})\Psi) = 0,$$

since

$$(A\overline{\varphi})\Psi \in \sum_{k=0}^{j-1} H_k.$$

It can be shown that H_n reduce $U(a, \Lambda)$, and therefore

$$U(a, \Lambda) = \bigoplus \sum_{j=0}^{\infty} U_j(a, \Lambda).$$

Let P_n be the orthoprojector onto H_n. Introduce the continuous operators

$$J_{jk}(\varphi) = P_{jA}(\varphi)P_k : H_k \to H_j \quad (j, k = 0, 1, ...); \qquad J_{jk}(\varphi) = 0 \quad (|j-k| > 1)$$

and construct the operator Jacobi matrix (3)

$$J(\varphi) = (J_{jk}(\varphi))_{j,k=0}^{\infty},$$

defined on the class D of finite sequences $\Phi = (\Phi_j)_{j=0}^{\infty}$ ($\Phi_j = P_j\Phi$) in the usual way:

$$(J(\varphi)\Phi)_j = \sum_{k=0}^{\infty} J_{jk}(\varphi)\Phi_k = (A(\varphi)\Phi)_j.$$

Leaving in $J(\varphi)$ only the elements respectively on the $(j+1, j)$-, (j, j)- and $(j, j+1)$-diagonals, and replacing the remaining ones by zeros, we obtain the operator matrices $J_+(\varphi)$, $J_0(\varphi)$, and $J_-(\varphi)$. Obviously,

$$J(\varphi) = J_+(\varphi) + J_0(\varphi) + J_-(\varphi) = J^*(\overline{\varphi}),$$

$$J_+^*(\varphi) = J_-(\overline{\varphi}), \qquad J_0^*(\varphi) = J_0(\overline{\varphi}).$$

The considerations outlined prove, in one direction, the following theorem.

Theorem 1. *A Hermitian scalar field is given if the following situation is given. A Hilbert space H is given, decomposed into an orthogonal sum of subspaces:*

$$H = \bigoplus \sum_{j=0}^{\infty} H_j, \qquad H \supset \Phi = (\Phi_j)_{j=0}^{\infty}, \quad \Phi_j \in H_j$$

(H_0 is one-dimensional and is spanned by the vacuum Ω) and a family of operator Jacobi matrices

$$J(\varphi) = (J_{jk}(\varphi))_{j,k=0}^{\infty} = J^*(\overline{\varphi}) \qquad (\varphi \in S(R^4)).$$

Here $J_{jk}(\varphi) : H_k \to H_j$ is continuous, and $\varphi \mapsto J_{jk}(\varphi)$ is linear and $J_{jk}(\varphi)$ depends weakly continuously on φ. It is required that polynomials in $J(\varphi)$, applied to Ω, give a dense set in H.

In each of the subspaces H_j there acts some unitary representation $U_j(a,\Lambda)$ of the group $\mathcal{P}_+^\uparrow$, with

$$U_j(a,\Lambda)J_{jk}(\varphi)U_k^{-1}(a,\Lambda) = J_{jk}(\varphi_{(a,\Lambda)}) \quad (j,k=0,1,\dots;\ (a,\Lambda)\in\mathcal{P}_+^\uparrow).$$

For $U_j(a,\Lambda)$, $j = 1,2,\dots$, axiom 3) is satisfied with ε independent of j, and, moreover,

$$0 \in \operatorname{supp} E_j(\Lambda).$$

It is required that for any $\varphi_1, \varphi_2 \in S(R^4)$ with compact and spacelike-separated supports the following equalities, equivalent to 4), hold:

$$[J_+(\varphi_1), J_+(\varphi_2)] = 0, \qquad [J_+(\varphi_1), J_0(\varphi_2)] + [J_0(\varphi_1), J_+(\varphi_2)] = 0, \tag{1}$$
$$[J_+(\varphi_1), J_-(\varphi_2)] + [J_0(\varphi_1), J_0(\varphi_2)] + [J_-(\varphi_1), J_+(\varphi_2)] = 0.$$

* As the authors have learned, a construction close to the one described was carried out by A. Wightman in an unpublished work.

3°. In what follows it will be seen that the decomposition of H in Theorem 1 is similar to the construction of Fock space ([4]), with the difference that the symmetries with respect to variables are of a more complicated character and the scalar product in each component is more general than the product in L_2. The matrices $J_+(\varphi)$ will play the role of creation operators—they generate H_n from the vacuum.

It is not difficult to prove that the c.l.s. $\{J_+(\varphi_n)\dots J_+(\varphi_1)\Omega;\ \varphi_1,\dots,\varphi_n \in S(R^4)\} = H_n$ $(n = 1,2,\dots)$. Form the $2n$-linear functional

$$\mathcal{E}_{2n}(\varphi_1(x_1),\dots,\varphi_n(x_n),\psi_1(y_1),\dots,\psi_n(y_n)) = (J_+(\psi_n)\dots J_+(\psi_1)\Omega,\ J_+(\bar\varphi_n)\dots J_+(\bar\varphi_1)\Omega)$$

on $S(R^4)$. As in the construction of the Wightman functionals, this functional can, by means of Schwartz' s kernel theorem, be extended by linearity and continuity to a functional $\mathcal{E}_{2n}$ on $S(R^{8n})$. Formally one may write:

$$\mathcal{E}_{2n}(x_1,\dots,x_n,y_1,\dots,y_n) = (J_+(y_n)\dots J_+(y_1)\Omega,\ J_+(x_n)\dots J_+(x_1)\Omega);$$

$\mathcal{E}_{2n}$ will be called the generating functional.

Theorem 2. *A Hermitian scalar field is given if the following situation obtains. A sequence of functionals $\mathcal{E}_{2n} \in S'(R^{8n})$ $(n = 1,2,\dots;\ \mathcal{E}_0 = 1)$ is given, which are positive definite:*

$$\mathcal{E}_{2n}(\bar u_n \otimes u_n) \ge 0 \qquad (u_n \in S(R^{4n}))$$

and are subordinate to one another in the sense that, for any $\varphi \in S(R^4)$,

$$\mathcal{E}_{2n+2}(\bar u_n \otimes \bar\varphi \otimes u_n \otimes \varphi) \le C_{n,\varphi}\mathcal{E}_{2n}(\bar u_n \otimes u_n)$$

$$(u_n \in S(R^{4n});\ C_{n,\varphi} < \infty;\ n = 0, 1, ...). \tag{2}$$

These functionals must be relativistically invariant:

$$\mathcal{E}_{2n}(u_{2n,(a,\Lambda)}) = \mathcal{E}_{2n}(u_{2n}) \quad ((a,\Lambda) \in \mathcal{P}_+^{\uparrow}).$$

In particular, they are translationally invariant and depend on differences of variables: there exist $E_{2n-1} \in S'(R^{4(2n-1)})$ *such that*

$$\mathcal{E}_{2n}(u_{2n}(x_1, \dots, y_n)) = E_{2n-1}(u_{2n}(x_2 - x_1, \dots, y_n - y_{n-1})).$$

The spectrality condition is written in the form: the Fourier transforms of the functional E_{2n-1} *are concentrated in*

$$\underbrace{V_{+,\varepsilon} \times \dots \times V_{+,\varepsilon}}_{2n-1} \quad (\varepsilon > 0;\ n = 1, 2, ...).$$

Local commutativity is expressed partially (namely, the first equality in (1)) by the relation: if $\operatorname{supp}\varphi_l$ *and* $\operatorname{supp}\varphi_{l+1}$ *are compact and spacelike separated* $(l = 1, \dots, n-1;\ n = 2, 3, ...)$*, then, for any remaining functions* $\varphi_j, \varphi_k \in S(R^4)$*, the expression*

$$\mathcal{E}_{2n}(\varphi_1 \otimes \dots \otimes \varphi_l \otimes \varphi_{l+1} \otimes \dots \otimes \varphi_n \otimes \psi_1 \otimes \dots \otimes \psi_n)$$

is symmetric with respect to φ_l, φ_{l+1}*; an analogous relation holds with respect to* $y_1, \dots, y_n$.

From the generating functionals one uniquely reconstructs the Hilbert space H and the "parts" of the field operators. Thus, on finite sequences

$$u = (u_j(x_1, \dots, x_j))_{j=0} \quad (u_0 \in \mathbb{C};\ u_j \in S(R^{4j}))$$

we introduce the quasi-scalar product

$$(u, v) = u_0 \bar{v}_0 + \sum_{j=1}^{\infty} \mathcal{E}_{2j}(\bar{v}_j \otimes u_j) \tag{3}$$

and perform the identification and completion. The space H_n is constructed by the same procedure applied to the n-th summand in (3);

$$H = \bigoplus_{j=0}^{\infty} H_j.$$

For each $\varphi \in S(R^4)$, on the finite sequences considered, a shift operator ("creation operator") is introduced

$$u = (u_0, u_1(x_1), u_2(x_1, x_2), ...) \to$$

$$\to (0, u_0\varphi(x_1), u_1(x_1)\varphi(x_2), u_2(x_1, x_2)\varphi(x_3), ...),$$

which, by virtue of (2), generates the operator matrix $J_+(\varphi)$ with only the $(j+1, j)$-th diagonal different from zero. On it stand the continuous operators

$$J_{j+1,j}(\varphi) : H_j \to H_{j+1},$$

generated by the correspondence $u_j \to u_j \otimes \varphi$ $(j = 0, 1, \ldots)$. Denote

$$J_-(\varphi) = J_+^*(\bar{\varphi}).$$

To specify the field it is necessary, in addition to the generating functionals, also to specify the (j, j)-diagonal operator matrix $J_0(\varphi)$ $(\varphi \in S(R^4))$,

diagonal elements are continuous operators $J_{jj}(\varphi) : H_j \to H_j$, depending weakly continuously on $\varphi \in S(R^4)$, and such that

$$(J_{jj}(\varphi)u_j)(a, \Lambda) = J_{jj}(\varphi_{(a,\Lambda)})u_j(a, \Lambda) \quad ((a, \Lambda) \in \mathcal{P}_+^\uparrow,\ u_j \in S(R^{4j}),\ j = 1, 2, \ldots),$$

and the last two equalities in (1) are satisfied. The field operators are the operators $A(\varphi)$ generated by the Jacobi matrix $J(\varphi) = J_+(\varphi) + J_0(\varphi) + J_-(\varphi)$ on finite sequences $\Phi = (\Phi_j)_{j=0}^\infty$ $(\Phi_j \in H_j)$.

4°. For the generating functionals one can, similarly to the Wightman functionals, construct and study generating functions $\mathcal{E}_{2n}(z_1, \ldots, z_{2n})$, analytic in the extended domain $\mathfrak{S}'_{2n}$ ([2]) and tending in the sense of $S'(R^{8n})$ to $\mathcal{E}_{2n}$ as $\operatorname{Im} z_j \to 0$. If $z_1, \ldots, z_{2n}$ are such that $(z_2 - z_1, \ldots, z_{2n} - z_{2n-1})$ is a Jost point, then $\mathcal{E}_{2n}(z_1, \ldots, z_{2n})$ is defined, holomorphic in a neighborhood of this point, and symmetric separately in the sets of variables $(z_1, \ldots, z_n)$ and $(z_{n+1}, \ldots, z_{2n})$.

5°. In terms of Theorem 1, the theory of the generalized free field ([5],[1],[2]) looks as follows. The decomposition of the space

$$H = \bigoplus_{j=0}^{\infty} H_j$$

and the unitary representations $U_n(a, \Lambda)$ are the same as in that theory. The Jacobi matrix is $J(\varphi) = J_+(\varphi) + J_-(\varphi)$ $(J_0(\varphi) = 0)$, where $J_+(\varphi)$ and $J_-(\varphi)$ are the usual creation and annihilation operators. To write the theory in terms of Theorem 2 one should write the generating functionals. It is easy to see that

$$\mathcal{E}_{2n}(u_n \otimes v_n) = \int_{R^{4n}} (\widetilde{Su_n})(p_1, \ldots, p_n) \times (\widetilde{Sv_n})(p_1, \ldots, p_n)\, d\rho(p_1) \cdots d\rho(p_n),$$

where S is the symmetrization operator in all variables, $\sim$ is the Fourier transform, and $d\rho(p)$ is the measure appearing in the definition of H_1. Concerning the possibility of constructing examples of fields on the basis of the transition described above to operator Jacobi matrices, see ([6],[7]).

Institute of Mathematics
Academy of Sciences of the Ukrainian SSR
Kiev

Received
18 III 1969

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Note: Figure translations are in progress. See original paper for figures.